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PERMISSIBLE PATTERNS OF PRIMES

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Abstract. A family of counting functions pertaining to prime k-tuples is introduced. These functions enumerate the permissible patterns of admissible prime k-tuples. The relationships of the functions are described, and properties of the functions are developed. The functions are then used to identify $y = 3159$ as the smallest value of y that contradicts the second Hardy-Littlewood conjecture which states $\pi(x+y) - \pi(x) \leq \pi(y)$. The functions are also used to determine the validity of the Hardy-Littlewood k-tuples conjecture.

A permissible pattern is the representation of the positions of the primes in an admissible prime tuple. The pattern '`.x.x...x.`' is a representation of prime tuples with the form $\{b+1, b+3, b+7\}$, one such tuple is $\{11, 13, 17\}$. The width of a pattern is the number of locations that are represented, the pattern '`.x.x...x.`' has a width of nine, or $w = 9$. The density of a pattern is the number of locations that represent primes, the pattern '`.x.x...x.`' consists of three locations that represent primes, or $k = 3$. The first and last locations of a permissible pattern are identified as the 'boundary locations', where the first location is the 'leading boundary' and the last location is the 'trailing boundary'.

The number of unique permissible patterns for any width can be enumerated using counting functions. A group of these counting functions are those dependant on the width w. These functions are $\rho\rho(w)$, $\rho\rho\text{f}(w)$, and $\rho\rho\text{b}(w)$. Another group of functions are those dependant on both the width w and the density k. These functions are $\rho(w,k)$, $\rho\text{f}(w,k)$, and $\rho\text{b}(w,k)$.

The counting function $\rho\rho(w)$ is defined as the number of permissible patterns when the width of the pattern is w. This function enumerates the number of unique admissible prime tuple variations that can exist in w consecutive integers. As an example, the 35 permissible patterns representing intervals of nine consecutive integers are shown below. Note that the empty pattern '`.........`' is also a countable pattern. See Table 1 for more values of $\rho\rho()$.

```
x........  x.x......  x...x....  x.x...x..  x.x.....x
.x.......  .x.x.....  .x...x...  .x.x...x.
..x......  ..x.x....  ..x...x..  ..x.x...x  x.....x.x
...x.....  ...x.x...  ...x...x.
....x....  ....x.x..  ....x...x  x...x.x..
.....x...  .....x.x.             .x...x.x.  x.x...x.x
......x..  ......x.x  x.....x..  ..x...x.x
.......x.             .x.....x.
........x  x.......x  ..x.....x             .........
```

The 35 permissible patterns enumerated by $\rho\rho(9)$

The counting function $\rho\rho\text{f}(w)$ is defined as the number of permissible patterns when the width of the pattern is w and the leading boundary represents a prime. This function enumerates the number of unique admissible prime tuple variations that can exist in w consecutive integers when the first integer is a prime. As an example, the 10 permissible patterns representing intervals of nine consecutive integers that start with a prime are shown below. See Table 2 for more values of $\rho\rho\text{f}()$.

```
x........  x.x......  x.x...x..  x.x...x.x
           x...x....  x.x.....x
           x.....x..  x...x.x..
           x.......x  x.....x.x
```

The 10 permissible patterns enumerated by $\rho\rho\text{f}(9)$

The counting function $\rho\rho\text{b}(w)$ is defined as the number of permissible patterns when the width of the pattern is w and both boundary locations represent primes. This function enumerates the number of unique admissible prime tuple variations that can exist in w consecutive integers when the first and last integer are prime. As an example, the 4 permissible patterns representing intervals of nine consecutive integers that start and end with a prime are shown below. See Table 3 for more values of $\rho\rho\text{b}()$.

```
x.......x  x.x.....x  x.x...x.x
           x.....x.x
```

The 4 permissible patterns enumerated by $\rho\rho\text{b}(9)$

The counting function $\rho(w, k)$ is defined as the number of permissible patterns when the width of the pattern is w and the density of the pattern is k. This function enumerates the number of unique admissible prime tuple variations of k primes that can exist in w consecutive integers. As an example, the 8 permissible patterns representing three primes in an interval of nine consecutive integers are shown below. See Table 1 for more values of $\rho()$.

```
x.x...x..  x...x.x..  x.x.....x
.x.x...x.  .x...x.x.  x.....x.x
..x.x...x  ..x...x.x
```

The 8 permissible patterns enumerated by $\rho(9, 3)$

The counting function $\rho\text{f}(w, k)$ is defined as the number of permissible patterns when the width of the pattern is w, the density of the pattern is k, and the leading boundary represents a prime. This function enumerates the number of unique admissible prime tuple variations of k primes that can exist in w consecutive integers when the first integer is a prime. As an example, the 4 permissible patterns representing three primes in an interval of nine consecutive integers that start with a prime are shown below. See Table 2 for more values of $\rho\text{f}()$.

```
x.x...x..  x.x.....x
x...x.x..  x.....x.x
```

The 4 permissible patterns enumerated by $\rho\text{f}(9, 3)$

The counting function $\rho\text{b}(w,k)$ is defined as the number of permissible patterns when the width of the pattern is w, the density of the pattern is k, and both boundary locations represent primes. This function enumerates the number of unique admissible prime tuple variations of k primes that can exist in w consecutive integers when the first and last integers are prime. As an example, the 2 permissible patterns representing three primes in an interval of nine consecutive integers that start and end with a prime are shown below. See Table 3 for more values of $\rho\text{b}()$.

`x.x.....x  x.....x.x`

The 2 permissible patterns enumerated by $\rho\text{b}(9,3)$

A permissible pattern consists of one or more locations, meaning the width is one or greater. No permissible pattern has a width of zero, therefore the counting functions $\rho\rho(w)$ and $\rho(w,k)$ are undefined for widths of zero.

$$\rho\rho(w) \text{ and } \rho(w,k) \qquad \text{are undefined for} \quad w \leq 0$$

The counting functions $\rho\rho\text{f}()$ and $\rho\text{f}()$ enumerate permissible patterns with a prime representation in the leading boundary location. A countable pattern must consist of one or more locations to have a leading boundary, meaning the width is one or greater. No permissible pattern with a leading boundary location has a width less than one, therefore the counting functions $\rho\rho\text{f}(w)$ and $\rho\text{f}(w,k)$ are undefined for widths less than one.

$$\rho\rho\text{f}(w) \text{ and } \rho\text{f}(w,k) \qquad \text{are undefined for} \quad w < 1$$

Also, a countable pattern must consist of a prime representation in the leading boundary, meaning the density is one or greater. No permissible pattern with a prime representation in the leading boundary has a density less than one, therefore the counting function $\rho\text{f}(w,k)$ is undefined for densities less than one.

$$\rho\text{f}(w,k) \qquad \text{is undefined for} \quad k < 1$$

The counting functions $\rho\rho\text{b}()$ and $\rho\text{b}()$ enumerate permissible patterns with a prime representation in both the leading and trailing boundary locations. A countable pattern must consist of two or more locations to have both a leading and trailing boundary, meaning the width is two or greater. No permissible pattern with a leading and trailing boundary location has a width less than two, therefore the counting functions $\rho\rho\text{b}(w)$ and $\rho\text{b}(w,k)$ are undefined for widths less than two.

$$\rho\rho\text{b}(w) \text{ and } \rho\text{b}(w,k) \qquad \text{are undefined for} \quad w < 2$$

Also, a countable pattern must consist of a prime representation in both the leading and trailing boundary, meaning the density is two or greater. No permissible pattern with a prime representation in both the leading and trailing boundary has a density less than two, therefore the counting function $\rho\text{b}(w,k)$ is undefined for densities less than two.

$$\rho\text{b}(w,k) \qquad \text{is undefined for} \quad k < 2$$

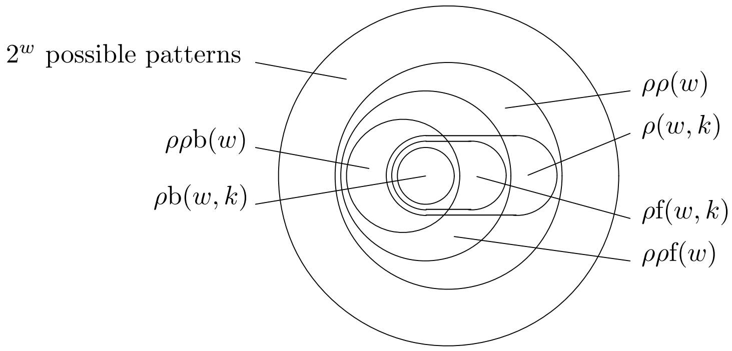
The counting functions $\rho()$, $\rho\mathrm{f}()$, and $\rho\mathrm{b}()$ enumerate permissible patterns with a width of w and a density of k. A countable pattern must consist of w locations, meaning the density can be no greater than w. No permissible pattern can have a density greater than the width, therefore no permissible patterns can exist when the density is greater than the width.

$$\rho(w,k) = 0 \qquad \text{when} \quad k > w$$
$$\rho\mathrm{f}(w,k) = 0 \qquad \text{when} \quad k > w$$
$$\rho\mathrm{b}(w,k) = 0 \qquad \text{when} \quad k > w$$

A permissible pattern that consists of only non-prime representations has a density of zero and is known as an *'empty pattern'*. Only one empty pattern exists for each width, therefore $\rho(w,0) = 1$ for widths of one or greater.

$$\rho(w,0) = 1 \qquad \text{for all} \quad w \geq 1$$

These counting functions can be viewed as sets of patterns in the universe of all possible patterns. As shown below the patterns enumerated by each counting function are a subset of all 2^w possible patterns for the width w. Also shown is the hierarchy of the counting functions. The counting function $\rho\rho(w)$ has the broadest range where each of the five other counting functions are proper subsets of $\rho\rho(w)$. The counting function $\rho\mathrm{b}(w,k)$ has the narrowest range because it is a proper subset in each of the five other counting functions.



The six counting functions viewed as sets

The counting function $\rho\rho(w)$ enumerates all permissible patterns with a width of w while the counting function $\rho(w,k)$ enumerates all permissible patterns with a width of w and a density of k. Every pattern enumerated by $\rho(w,k)$ is also enumerated by $\rho\rho(w)$.

$$\rho\rho(w) = \rho(w,w) + \rho(w,w-1) + \ldots + \rho(w,1) + \rho(w,0)$$

$$\rho\rho(w) = \sum_{j=0}^{w} \rho(w,j) \qquad \text{for all} \quad w > 0 \tag{1}$$

The counting function $\rho\rho\text{f}(w)$ enumerates all permissible patterns with a width of w and a prime representation in the leading boundary location while the counting function $\rho\text{f}(w,k)$ enumerates all permissible patterns with a width of w, a density of k, and a prime representation in the leading boundary location. Every pattern enumerated by $\rho\text{f}(w,k)$ is also enumerated by $\rho\rho\text{f}(w)$.

$$\rho\rho\text{f}(w) = \rho\text{f}(w,w) + \rho\text{f}(w,w-1) + \ldots + \rho\text{f}(w,2) + \rho\text{f}(w,1)$$

$$\rho\rho\text{f}(w) = \sum_{j=1}^{w} \rho\text{f}(w,j) \qquad \text{for all} \quad w \geq 1 \tag{2}$$

The counting function $\rho\rho\text{b}(w)$ enumerates all permissible patterns with a width of w and prime representations in both boundary locations while the counting function $\rho\text{b}(w,k)$ enumerates all permissible patterns with a width of w, a density of k, and prime representations in both boundary locations. Every pattern enumerated by $\rho\text{b}(w,k)$ is also enumerated by $\rho\rho\text{b}(w)$.

$$\rho\rho\text{b}(w) = \rho\text{b}(w,w) + \rho\text{b}(w,w-1) + \ldots + \rho\text{b}(w,3) + \rho\text{b}(w,2)$$

$$\rho\rho\text{b}(w) = \sum_{j=2}^{w} \rho\text{b}(w,j) \qquad \text{for all} \quad w \geq 2 \tag{3}$$

The summations created so far have been based on varying the number of prime representations in the patterns. Summations can also be created by varying the width of the patterns. An operation for manipulating a permissible pattern is trimming. Trimming shortens a permissible pattern by truncating either boundary location. The result of this operation is a pattern that represents an admissible prime tuple, therefore the resulting pattern is a permissible pattern. A consequence of the trimming operation is any contiguous sequence of locations within a permissible pattern is itself a permissible pattern.

The counting function $\rho\rho(w)$ enumerates all permissible patterns with a width of w. These patterns can be divided into two groups based on the leading boundary. The first group consists of patterns with a prime representation in the leading boundary and the second group consists of patterns with a non-prime representation in the leading boundary. The first group is equivalent to the patterns enumerated by $\rho\rho\text{f}(w)$. Trimming the non-prime representation in the leading boundary location from every pattern in the second group of patterns produces the patterns enumerated by $\rho\rho(w-1)$.

$$\rho\rho(w) = \rho\rho\text{f}(w) + \rho\rho(w-1) \qquad \text{for all} \quad w \geq 1$$

This division into two groups can continue for the $\rho\rho()$ term and every successive $\rho\rho()$ term until the width is 1. The two possible patterns with a width of one are `x` and `.`. Both patterns represent admissible prime tuples, therefore $\rho\rho(1) = 2$ and $\rho\rho\text{f}(1) = 1$.

$$\rho\rho(w) = \rho\rho\text{f}(w) + \rho\rho\text{f}(w-1) + \ldots + \rho\rho\text{f}(2) + \rho\rho(1)$$
$$\rho\rho(w) = \rho\rho\text{f}(w) + \rho\rho\text{f}(w-1) + \ldots + \rho\rho\text{f}(2) + \rho\rho\text{f}(1) + 1$$

The counting function $\rho\rho()$ can be expressed as a summation of counting function $\rho\rho\text{f}()$ values. It should be noted that the value of $\rho\rho\text{f}(w) \geq 1$ for all $w \geq 1$, therefore $\rho\rho(w+1) > \rho\rho(w)$, meaning the counting function $\rho\rho()$ is strictly increasing.

$$\rho\rho(w) = 1 + \sum_{i=1}^{w} \rho\rho\text{f}(i) \qquad \text{for all} \quad w \geq 1 \tag{4}$$

The counting function $\rho\rho\text{f}(w)$ enumerates all permissible patterns with a width of w and a prime representation in the leading boundary. These patterns can be divided into two groups based on the trailing boundary. The first group consists of patterns with a prime representation in the trailing boundary and the second group consists of patterns with a non-prime representation in the trailing boundary. The first group is equivalent to the patterns enumerated by $\rho\rho\text{b}(w)$. Trimming the non-prime representation in the trailing boundary location from every pattern in the second group of patterns produces the patterns enumerated by $\rho\rho\text{f}(w-1)$.

$$\rho\rho\text{f}(w) = \rho\rho\text{b}(w) + \rho\rho\text{f}(w-1) \qquad \text{for all} \quad w \geq 2$$

This division into two groups can continue for the $\rho\rho\text{f}()$ term and every successive $\rho\rho\text{f}()$ term until the width is 2. The four possible patterns with a width of two are '`..`', '`x.`', '`.x`' and '`xx`'. The pattern '`xx`' does not represent an admissible prime tuple, therefore $\rho\rho\text{f}(2) = 1$ and $\rho\rho\text{b}(2) = 0$.

$$\rho\rho\text{f}(w) = \rho\rho\text{b}(w) + \rho\rho\text{b}(w-1) + \ldots + \rho\rho\text{b}(3) + \rho\rho\text{f}(2)$$
$$\rho\rho\text{f}(w) = \rho\rho\text{b}(w) + \rho\rho\text{b}(w-1) + \ldots + \rho\rho\text{b}(3) + \rho\rho\text{b}(2) + 1$$

The counting function $\rho\rho\text{f}()$ can be expressed as a summation of counting function $\rho\rho\text{b}()$ values. It should be noted that the value of $\rho\rho\text{b}(w) \geq 0$ for all $w \geq 2$, therefore $\rho\rho\text{f}(w+1) \geq \rho\rho\text{f}(w)$, meaning the counting function $\rho\rho\text{f}()$ is weakly increasing.

$$\rho\rho\text{f}(w) = 1 + \sum_{i=2}^{w} \rho\rho\text{b}(i) \qquad \text{for all} \quad w \geq 2 \tag{5}$$

Similar summations can be created for the counting functions that are dependent on both w and k. The counting function $\rho(w,k)$ enumerates all permissible patterns with a width of w and a density of k. These patterns can be divided into two groups based on the leading boundary. The first group consists of patterns with a prime representation in the leading boundary and the second group consists of patterns with a non-prime representation in the leading boundary. The first group is equivalent to the patterns enumerated by $\rho\text{f}(w,k)$. Trimming the non-prime representation in the leading boundary location from every pattern in the second group produces the patterns enumerated by $\rho(w-1,k)$.

$$\rho(w,k) = \rho\text{f}(w,k) + \rho(w-1,k) \qquad \text{for all} \quad 1 \leq k \leq w$$

This division into two groups can continue for the $\rho()$ term and every successive $\rho()$ term until the width is k. When w equals k all locations of the pattern must be prime representations, therefore the leading boundary is a prime representation and the enumeration of $\rho(k,k)$ equals the enumeration of $\rho\mathrm{f}(k,k)$.

$$\rho(w,k) = \rho\mathrm{f}(w,k) + \rho\mathrm{f}(w-1,k) + \ldots + \rho\mathrm{f}(k+1,k) + \rho(k,k)$$
$$\rho(w,k) = \rho\mathrm{f}(w,k) + \rho\mathrm{f}(w-1,k) + \ldots + \rho\mathrm{f}(k+1,k) + \rho\mathrm{f}(k,k)$$

The counting function $\rho()$ can be expressed as a summation of counting function $\rho\mathrm{f}()$ values. The counting function $\rho()$ is a strictly increasing function.

$$\rho(w,k) = \sum_{i=k}^{w} \rho\mathrm{f}(i,k) \qquad \text{for all} \quad 1 \le k \le w \tag{6}$$

The counting function $\rho\mathrm{f}(w,k)$ enumerates all permissible patterns with a width of w, a density of k, and the leading boundary location is a prime representation. These patterns can be divided into two groups based on the trailing boundary. The first group consists of patterns with a prime representation in the trailing boundary and the second group consists of patterns with a non-prime representation in the trailing boundary. The first group is equivalent to the patterns enumerated by $\rho\mathrm{b}(w,k)$. Trimming the non-prime representation from every pattern in the second group produces the patterns enumerated by $\rho\mathrm{f}(w-1,k)$.

$$\rho\mathrm{f}(w,k) = \rho\mathrm{b}(w,k) + \rho\mathrm{f}(w-1,k) \qquad \text{for all} \quad 2 \le k \le w$$

This division into two groups can continue for the $\rho\mathrm{f}()$ term and every successive $\rho\mathrm{f}()$ term until the width is k. When w equals k all locations of the pattern must be prime representations, therefore the trailing boundary is a prime representation and the enumeration of $\rho\mathrm{f}(k,k)$ equals the enumeration of $\rho\mathrm{b}(k,k)$.

$$\rho\mathrm{f}(w,k) = \rho\mathrm{b}(w,k) + \rho\mathrm{b}(w-1,k) + \ldots + \rho\mathrm{b}(k+1,k) + \rho\mathrm{f}(k,k)$$
$$\rho\mathrm{f}(w,k) = \rho\mathrm{b}(w,k) + \rho\mathrm{b}(w-1,k) + \ldots + \rho\mathrm{b}(k+1,k) + \rho\mathrm{b}(k,k)$$

The counting function $\rho\mathrm{f}()$ can be expressed as a summation of counting function $\rho\mathrm{b}()$ values. The counting function $\rho\mathrm{f}()$ is a weakly increasing function.

$$\rho\mathrm{f}(w,k) = \sum_{i=k}^{w} \rho\mathrm{b}(i,k) \qquad \text{for all} \quad 2 \le k \le w \tag{7}$$

Additional summations can be created by substituting. The counting function $\rho\rho()$ can be expressed as a summation of counting function $\rho\rho\mathrm{b}()$ values by substituting equation (5) into equation (4).

$$\rho\rho(w) = 1 + \rho\rho\mathrm{f}(1) + \sum_{i=2}^{w} \left(1 + \sum_{j=2}^{i} \rho\rho\mathrm{b}(j)\right)$$

$$\rho\rho(w) = 1 + w + \sum_{i=2}^{w} (w+1-i)\rho\rho\mathrm{b}(i) \qquad \text{for all} \quad w \ge 2 \tag{8}$$

The counting function $\rho()$ can be expressed as a summation of counting function $\rho\mathrm{b}()$ values by substituting equation (7) into equation (6).

$$\rho(w,k) = \sum_{i=k}^{w}\left(\sum_{ii=k}^{i}\rho\mathrm{b}(ii,k)\right)$$

$$\rho(w,k) = \sum_{i=k}^{w}\sum_{ii=k}^{i}\rho\mathrm{b}(ii,k)$$

$$\rho(w,k) = \sum_{i=k}^{w}(w+1-i)\rho\mathrm{b}(i,k) \qquad \text{for all} \quad 2 \le k \le w \tag{9}$$

The counting function $\rho\rho()$ can be expressed as a summation of counting function $\rho\mathrm{f}()$ values by substituting equation (2) into equation (4).

$$\rho\rho(w) = 1 + \sum_{i=1}^{w}\left(\sum_{j=1}^{i}\rho\mathrm{f}(i,j)\right)$$

$$\rho\rho(w) = 1 + \sum_{i=1}^{w}\sum_{j=1}^{i}\rho\mathrm{f}(i,j) \qquad \text{for all} \quad w \ge 1 \tag{10}$$

The counting function $\rho\rho\mathrm{f}()$ can be expressed as a summation of counting function $\rho\mathrm{b}()$ values by substituting equation (3) into equation (5).

$$\rho\rho\mathrm{f}(w) = 1 + \sum_{i=2}^{w}\left(\sum_{j=2}^{i}\rho\mathrm{b}(i,j)\right)$$

$$\rho\rho\mathrm{f}(w) = 1 + \sum_{i=2}^{w}\sum_{j=2}^{i}\rho\mathrm{b}(i,j) \qquad \text{for all} \quad w \ge 2 \tag{11}$$

The counting function $\rho\rho()$ can be expressed as a summation of counting function $\rho\mathrm{b}()$ values by substituting equation (3) into equation (8).

$$\rho\rho(w) = 1 + w + \sum_{i=2}^{w}(w+1-i)\left(\sum_{j=2}^{i}\rho\mathrm{b}(i,j)\right)$$

$$\rho\rho(w) = 1 + w + \sum_{i=2}^{w}\sum_{j=2}^{i}(w+1-i)\rho\mathrm{b}(i,j) \qquad \text{for all} \quad w \ge 2 \tag{12}$$

As shown in equations (3), (7), (9), (11), and (12), summations of counting function $\rho\mathrm{b}()$ values can be used to express each of the other five counting functions. The counting function $\rho\mathrm{b}()$ is the core function and requires further investigation. An admissible prime tuple that begins and ends with a prime can only exist in an odd number of consecutive integers, otherwise either the beginning or ending number would be divisible by two and could not be a prime. Every pattern that is countable by the $\rho\mathrm{b}()$ function must have prime representations in both boundary locations, therefore no permissible pattern with prime representations in both boundary locations can exist in a width that is even.

$$\rho\mathrm{b}(2x,k)=0 \qquad \text{for all} \quad x\geq 1 \text{ and } k\geq 2$$

The patterns that are countable by the $\rho\mathrm{b}()$ function have widths that are odd. Every pattern that is countable by the $\rho\mathrm{b}()$ function must have a density of two or more. When the density is two, the prime representations in both boundary locations are the only prime representations in the pattern. Only one countable pattern with a density of two can exist for each odd width.

$$\rho\mathrm{b}(2x+1,2)=1 \qquad \text{for all} \quad x\geq 1$$

The case of $k=2$ is generalized as

$$\text{for} \quad w\geq 2, \qquad \rho\mathrm{b}(w,2)=\begin{cases} 0 & \text{when } w \text{ is even} \\ 1 & \text{when } w \text{ is odd}\end{cases}$$

Trimming the trailing boundary location from a pattern enumerated by $\rho\mathrm{b}(w,k)$ results in a pattern enumerated by $\rho\mathrm{f}(w-1,k-1)$. If the trailing boundary location of the resulting pattern is a prime representation then this resulting pattern is also enumerated by $\rho\mathrm{b}(w-1,k-1)$. If the trailing boundary location of the resulting pattern is a non-prime representation continue trimming the trailing boundary location from the pattern until the trailing boundary location is a prime representation. The final resulting pattern is a pattern enumerated by both $\rho\mathrm{f}(w-a,k-1)$ and $\rho\mathrm{b}(w-a,k-1)$, where a is the number of times the trailing boundary location was trimmed. The number of patterns enumerated by $\rho\mathrm{b}(w,k)$ must be less than or equal to the sum of the number of patterns enumerated by $\rho\mathrm{b}(i,k-1)$ for every $i<w$, provided the initial w is greater than 2.

$$\rho\mathrm{b}(w,k)\leq\sum_{i=2}^{w-1}\rho\mathrm{b}(i,k-1) \qquad \text{for all} \quad w>2$$

This summation can be revised to sum over only odd widths because $\rho\mathrm{b}()$ is equal to zero for all even widths.

$$\rho\mathrm{b}(2x+1,k)\leq\sum_{i=1}^{x-1}\rho\mathrm{b}(2i+1,k-1) \qquad \text{for all} \quad x\geq 1$$

A generalization is created when this same summation is applied to all of the $\rho\mathrm{b}(2i+1,k-1)$ terms in the original summation.

$$\rho\mathrm{b}(2x+1,k)\leq\sum_{i=1}^{x-k+n}\binom{x-1-i}{k-n-1}\rho\mathrm{b}(2i+1,n) \qquad \text{when} \quad 2\leq n<k \tag{13}$$

An upper bound is established by setting $n = 2$ in equation (13), thereby allowing the previously established equality of $\rho\mathrm{b}(2x+1,2) = 1$ to be used.

$$\rho\mathrm{b}(2x+1,k) \leq \sum_{i=1}^{x-k+2} \binom{x-1-i}{k-3}$$
$$\rho\mathrm{b}(2x+1,k) \leq \binom{x-1}{k-2}$$

This upper bound of $\binom{x-1}{k-2}$ for $\rho\mathrm{b}(2x+1,k)$ is simply the number of ways to select a subset of $k-2$ elements from a set of $x-1$ elements. The patterns enumerated by $\rho\mathrm{b}(2x+1,k)$ are of an odd width with a prime representation in both the leading and trailing boundaries. When the width of a pattern is $2x+1$ there are x locations that must be non-prime representations of the even numbers in the corresponding admissible prime tuple. The remaining $x+1$ locations are available locations for prime representations. Two of these locations are the leading and trailing boundaries which already are prime representations, leaving $x-1$ possible locations for prime representations. The patterns enumerated by $\rho\mathrm{b}(2x+1,k)$ must also contain k prime representations. Again, two of these representations are in the leading and trailing boundaries, leaving $k-2$ prime representations to be distributed throughout the $x-1$ available locations in the pattern. The upper bound of $\rho\mathrm{b}()$ can be greatly improved by using larger values of n in equation (13). More information about the character of the counting function $\rho\mathrm{b}()$ is required to use a larger value of n.

When all factors of an integer in an admissible prime tuple that is enumerated by $\rho\mathrm{b}(w,k)$ are greater than k the location in the corresponding permissible pattern is a 'possible' location for a prime representation. An example of a possible location is in the admissible prime tuple $\{31,37,41,43\}$ that is enumerated by $\rho\mathrm{b}(13,4)$. This integer sequence contains the integer 35 that has the factors 5 and 7. Both factors are greater than the density of the enumerating function. The location in the permissible pattern representation that corresponds to 35 in the integer sequence is a possible location for a prime representation. The locations of the prime representations in the original permissible pattern are also possible locations for prime representations.

```
                              {31,37,41,43}  Admissible prime tuple
31 32 33 34 35 36 37 38 39 40 41 42 43  Integers in tuple
 x  .  .  .  .  .  x  .  .  .  x  .  x  Prime locations
31  2  3  2  5  2 37  2  3  2 41  2 43  Smallest factor
 s  .  .  .  s  .  s  .  .  .  s  .  s  Possible locations
```

If the number of possible locations exceeds the number of prime representations in a permissible pattern additional permissible patterns can be created. Every combination of k or fewer possible locations creates a valid permissible pattern. Permissible patterns created from k possible locations which include both boundary locations are unique permissible patterns enumerated by $\rho\mathrm{b}(w,k)$. The quantity of unique permissible patterns that can be created is $\binom{s-2}{k-2}$ where s represents the number of possible locations. If the quantity of possible locations equals the quantity of prime representations the binomial equals one, being the original permissible pattern.

The product of the primes less than or equal to k is commonly known as a primorial, hereby denoted as $\mathbb{P}_k$. Applying the 'Sieve of Eratosthenes' on the integers 1 through $\mathbb{P}_k$ with all the primes less than or equal to k exposes the integers having all factors greater than k. The corresponding locations of these integers are possible locations for prime representations that can be used to create permissible patterns with a density of k.

$\mathbb{P}_6 = 2{\cdot}3{\cdot}5 = 30$ locations	s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s
remove every $p_1 = 2$nd integer	s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2 s 2
remove every $p_2 = 3$rd integer	s . 3 . s - s . 3 . s - s . 3 . s - s . 3 . s - s . 3 . s -
remove every $p_3 = 5$th integer	s . . . 5 . s . . - s . s . - . s . s - . . s . 5 . . . s -
8 possible locations	s . . . . . s . . . s . s . . . s . s . . . s . . . . . s .

The number of integers exposed in the first $\mathbb{P}_k$ integers is calculated as the product of $(p_i - 1)$ for all primes p_i that are less than or equal to k. This product is denoted as $\mathbb{Q}_k$. Initial values and equations for the products $\mathbb{P}_k$ and $\mathbb{Q}_k$ are shown below.

k	$\pi(k)$	$p_{\pi(k)}$	$\mathbb{P}_k$	$p_{\pi(k)}-1$	$\mathbb{Q}_k$
2	1	2	2	1	1
3	2	3	6	2	2
4	2	3	6	2	2
5	3	5	30	4	8
6	3	5	30	4	8
7	4	7	210	6	48
8	4	7	210	6	48
9	4	7	210	6	48
10	4	7	210	6	48
11	5	11	2310	10	480
12	5	11	2310	10	480
13	6	13	30030	12	5760

Values and Equations of $\mathbb{P}_k$ and $\mathbb{Q}_k$

$$\mathbb{P}_k = \prod_{i=1}^{\pi(k)} p_i$$

$$\mathbb{Q}_k = \prod_{i=1}^{\pi(k)} (p_i - 1)$$

$\pi()$ is the prime counting function and p_i is the ith prime number

Using Euclid's argument about the infinitude of the primes, the integer after the primorial $\mathbb{P}_k$ is not divisible by any prime less than or equal to k, and is a possible location for a prime representation in permissible patterns enumerated by $\rho\mathrm{b}(\mathbb{P}_k + 1, k)$. Since the number of possible locations in a permissible pattern of $\mathbb{P}_k$ locations is $\mathbb{Q}_k$, the additional possible location at $\mathbb{P}_k + 1$ causes the number of possible locations in permissible patterns with a width of $\mathbb{P}_k + 1$ to be $\mathbb{Q}_k + 1$. A lower bound for the number of permissible patterns enumerated by $\rho\mathrm{b}(\mathbb{P}_k + 1, k)$ is immediately revealed.

$$\binom{\mathbb{Q}_k - 1}{k - 2} \quad \leq \quad \rho\mathrm{b}(\mathbb{P}_k + 1, k)$$

Consider a prime representation in a permissible pattern enumerated by $\rho\text{b}(w, k)$, the integer in the corresponding admissible prime tuple has a unique set of residues for each prime less than or equal to k. When $\mathbb{P}_k$ is added to the initial integer the resulting sum has the same set of residues for each prime less than or equal to k. Let $a_1, a_2, \ldots a_{r+1}$ be the possible locations of prime representations for a permissible pattern enumerated by $\rho\text{b}(\mathbb{P}_k + 1, k)$ where $r + 1$ is the number of possible locations, then $a_1, a_2, \ldots a_{r+1}$ and $a_1 + \mathbb{P}_k, a_2 + \mathbb{P}_k, \ldots a_{r+1} + \mathbb{P}_k$ are possible locations of prime representations for permissible patterns with a density of k. As shown, when a_i is a possible location, the location at $a_i + \mathbb{P}_k$ is also a possible location. Every permissible pattern enumerated by $\rho\text{b}(\mathbb{P}_k + 1, k)$ is contained in the sequence of possible locations from 1 to $2\mathbb{P}_k$. There are $\mathbb{Q}_k$ unique sequences of possible locations that are $\mathbb{P}_k + 1$ wide with each sequence starting at one of the possible locations from 1 through $\mathbb{P}_k$ and ending with one of the possible locations from $\mathbb{P}_k + 1$ through $2\mathbb{P}_k$. This leads to an upper bound for $\rho\text{b}(\mathbb{P}_k + 1, k)$.

$$\rho\text{b}(\mathbb{P}_k + 1, k) \quad \leq \quad \mathbb{Q}_k \binom{\mathbb{Q}_k - 1}{k - 2}$$

Using the same argument as above the sequence of possible locations is shown to repeat for every sequence of $\mathbb{P}_k$ locations. The lower and upper bounds are rewritten to reflect this multiple of $\mathbb{P}_k$ by including the multiple of $\mathbb{Q}_k$ in the binomials.

$$\binom{x\mathbb{Q}_k - 1}{k - 2} \quad \leq \quad \rho\text{b}(x\mathbb{P}_k + 1, k) \quad \leq \quad \mathbb{Q}_k \binom{x\mathbb{Q}_k - 1}{k - 2} \tag{14}$$

The exact quantity of permissible patterns enumerated by $\rho\text{b}(x\mathbb{P}_k + 1, k)$ is the upper bound given in equation (14) minus any duplicate patterns generated. The first step is to identify the $\mathbb{Q}_k$ unique possible location sequences. The $\mathbb{Q}_k$ sequences are created by performing the 'Sieve of Eratosthenes' on the integers 1 through $(x + 1)\mathbb{P}_k$ with all the primes less than or equal to k. The exposed integers correspond to possible locations for prime representations in permissible patterns with a density of k and a width of $(x + 1)\mathbb{P}_k$. Each sequence of $x\mathbb{P}_k + 1$ sieved locations that starts with a possible location contains $x\mathbb{Q}_k + 1$ possible locations.

Possible locations corresponding to the integers
1 through 60 with all factors greater than 5

```
s.....s...s.s...s.s...s.....s.s.....s...s.s...s.s...s.....s.
```

8 sequences of 31 consecutive locations

```
s.....s...s.s...s.s...s.....s.s
      s...s.s...s.s...s.....s.s.....s
          s.s...s.s...s.....s.s.....s...s
            s...s.s...s.....s.s.....s...s.s
                s.s...s.....s.s.....s...s.s...s
                  s...s.....s.s.....s...s.s...s.s
                      s.....s.s.....s...s.s...s.s...s
                            s.s.....s...s.s...s.s...s.....s
```

The diagram above displays sieving $(x + 1)\mathbb{P}_k$ integers to create the $\mathbb{Q}_k$ possible location sequences for the permissible patterns enumerated by $\rho\text{b}(x\mathbb{P}_k + 1, k)$ when

$k = 5$ and $x = 1$. Here the value of $\mathbb{Q}_k$ is 8, the value of $\mathbb{P}_k$ is 30, and the value of $x\mathbb{P}_k + 1$ is 31. These possible location sequences and the permissible patterns enumerated by $\rho\text{b}(31, 5)$ are used as examples in the following text.

The eight possible location sequences for $\rho\text{b}(31, 5)$ are identified as $\mathbf{A}_1$ through $\mathbf{A}_8$ and each sequence contains nine possible locations. The upper bound of $\rho\text{b}(31, 5)$ is calculated as $8\binom{9-2}{5-2}$ which equals 280.

$\mathbf{A}_1$	s.....s...s.s...s.s...s.....s.s
$\mathbf{A}_2$	s...s.s...s.s...s.....s.s.....s
$\mathbf{A}_3$	s.s...s.s...s.....s.s.....s...s
$\mathbf{A}_4$	s...s.s...s.....s.s.....s...s.s
$\mathbf{A}_5$	s.s...s.....s.s.....s...s.s...s
$\mathbf{A}_6$	s...s.....s.s.....s...s.s...s.s
$\mathbf{A}_7$	s.....s.s.....s...s.s...s.s...s
$\mathbf{A}_8$	s.s.....s...s.s...s.s...s.....s

The next step is to remove any permissible patterns that are duplicated in the upper bound. Duplicate permissible patterns exist when two or more possible location sequences create the same permissible pattern. The number of duplicate patterns generated by a combination of possible location sequences is determined by the quantity of common possible locations in the sequences.

$m = 2$	$\mathbf{A}_1$	s.....s...s.s...s.s...s.....s.s	
	$\mathbf{A}_2$	s...s.s...s.s...s.....s.s.....s	
	$\mathbf{A}_1 \bigcap \mathbf{A}_2$	s.....s...s.s...s.....s.......s	$n = 7$ common locations
$m = 3$	$\mathbf{A}_1$	s.....s...s.s...s.s...s.....s.s	
	$\mathbf{A}_2$	s...s.s...s.s...s.....s.s.....s	
	$\mathbf{A}_4$	s...s.s...s.....s.s.....s...s.s	
	$\mathbf{A}_1 \bigcap \mathbf{A}_2 \bigcap \mathbf{A}_4$	s.....s...s.....s.............s	$n = 5$ common locations

Examples of common possible locations

There are $\binom{\mathbb{Q}_k}{m}$ combinations of m different possible location sequences and each combination contains $n_{m,i}$ common possible locations. The number of duplicated permissible patterns for each combination of different possible location sequences is $\binom{n_{m,i}-2}{k-2}$. When $m = 1$ the quantity of common possible locations is just the number of possible locations in each sequence, or $n_{1,i} = x\mathbb{Q}_k + 1$ for $i = 1$ to $\mathbb{Q}_k$. This corresponds directly to the current upper bound.

explain the subcript i

$$\rho\text{b}(x\mathbb{P}_k + 1, k) \quad \leq \quad \mathbb{Q}_k \binom{x\mathbb{Q}_k - 1}{k-2} \quad = \quad \sum_{i=1}^{\binom{\mathbb{Q}_k}{1}} \binom{n_{1,i} - 2}{k-2}$$

Since this is an upper bound there may be duplicate permissible patterns counted by the summation of the binomials. The number of duplicate permissible patterns is identified by determining the quantity of common possible locations for every combination of two different possible location sequences and then calculate the number of permissible patterns the common possible locations can create. There are $\binom{\mathbb{Q}_k}{2}$

again explain the subcript i

combinations of two different possible location sequences and each combination contains $n_{2,i}$ common possible locations. Subtracting the permissible patterns created when $m = 2$ from the current upper bound generates a new lower bound.

$$\rho\mathrm{b}(x\mathbb{P}_k + 1, k) \quad \geq \quad \sum_{i=1}^{\binom{\mathbb{Q}_k}{1}} \binom{n_{1,i} - 2}{k - 2} - \sum_{i=1}^{\binom{\mathbb{Q}_k}{2}} \binom{n_{2,i} - 2}{k - 2}$$

Similar to the summation of the binomials counting the permissible patterns when $m = 1$, the quantity of permissible patterns removed by the summation of the binomials when $m = 2$ may also include duplicate permissible patterns. Setting $m = 3$ generates the number of permissible patterns created by the common possible locations of three different possible location sequences. Adding this summation of the binomials to the current lower bound generates an improved upper bound.

$$\rho\mathrm{b}(x\mathbb{P}_k + 1, k) \quad \leq \quad \sum_{i=1}^{\binom{\mathbb{Q}_k}{1}} \binom{n_{1,i} - 2}{k - 2} - \sum_{i=1}^{\binom{\mathbb{Q}_k}{2}} \binom{n_{2,i} - 2}{k - 2} + \sum_{i=1}^{\binom{\mathbb{Q}_k}{3}} \binom{n_{3,i} - 2}{k - 2}$$

An exact value for $\rho\mathrm{b}(x\mathbb{P}_k+1, k)$ is generated when this procedure is repeated for every value of m through $m = \mathbb{Q}_k$. When m is even the summation of the binomials is subtracted from the current upper bound and when m is odd the summation of the binomials is added to the current lower bound. It is to be noted that $\mathbb{Q}_k$ is even for all values of k greater than 2.

$$\rho\mathrm{b}(x\mathbb{P}_k + 1, k) \quad = \quad \sum_{i=1}^{\binom{\mathbb{Q}_k}{1}} \binom{n_{1,i} - 2}{k - 2} - \sum_{i=1}^{\binom{\mathbb{Q}_k}{2}} \binom{n_{2,i} - 2}{k - 2} + \cdots - \sum_{i=1}^{\binom{\mathbb{Q}_k}{\mathbb{Q}_k}} \binom{n_{\mathbb{Q}_k,i} - 2}{k - 2}$$

A double summation is made by reformatting this equation to account for the alternating sign of the summation of the binomials.

$$\rho\mathrm{b}(x\mathbb{P}_k + 1, k) \quad = \quad \sum_{j=1}^{\mathbb{Q}_k} \sum_{i=1}^{\binom{\mathbb{Q}_k}{j}} (-1)^{j-1} \binom{n_{j,i} - 2}{k - 2} \tag{15}$$

The quantity of common possible locations, $n_{m,i}$, for each combination of m different possible location sequences is a variable to be evaluated. The value of $n_{m,i}$ is investigated by manually tallying the number of common possible locations for every combination of possible location sequences of $\rho\mathrm{b}(31, 5)$.

$k = 5$ m	$\binom{\mathbb{Q}_5}{m}$	9	8	7	6	5	4	3	2
1	8	8							
2	28			12		4	12		
3	56					8	24	24	
4	70						6	50	14
5	56							24	32
6	28							4	24
7	8								8
8	1								1

Common possible location counts for $\rho\mathrm{b}(31, 5)$

The first column is the number of possible location sequences in a combination and the second column is the number of different combinations that exist. The third row displays the 56 combinations of 3 different possible location sequences. Of these, 8 combinations have 5 common possible locations, 24 combinations have 4 common possible locations, and 24 combinations have 3 common possible locations. The number of duplicate permissible patterns identified by the values given in the third row is 8.

$$8\cdot\binom{5-2}{5-2}+24\cdot\binom{4-2}{5-2}+24\cdot\binom{3-2}{5-2} \quad = \quad 8+0+0 \quad = \quad 8$$

The table of common possible location counts for permissible patterns of $\rho\mathrm{b}(31,5)$ can be modified to determine the common possible location counts for permissible patterns of $\rho\mathrm{b}(30x+1,5)$. The cyclic nature of the possible location sequences dictate that the common possible locations are also cyclic and exist in the same quantity for every width of $\mathbb{P}_k$ locations in the sequence combination. The common possible location counts remain the same since there is no overlap of common possible locations.

$k=5$ m	$\binom{\mathbb{Q}_5}{m}$	8x+1	7x+1	6x+1	5x+1	4x+1	3x+1	2x+1	x+1
1	8	8							
2	28			12		4	12		
3	56					8	24	24	
4	70						6	50	14
5	56							24	32
6	28							4	24
7	8								8
8	1								1

Common possible location counts for $\rho\mathrm{b}(30x+1,5)$

The table of common possible location counts is modified to display the number of duplicate permissible patterns by multiplying the common possible location counts and the binomial $\binom{n_{m,i}-2}{k-2}$ that describes the number of duplicated permissible patterns for a combination of possible location sequences. Also, dependant on the quantity of sequences in the combination, the sign of row is changed to account for the addition or subtraction of the duplicate patterns.

Upper Bound	$8\binom{8x-1}{3}$					
$-\bigcap$ of 2		$-12\binom{6x-1}{3}$	$-4\binom{4x-1}{3}$	$-12\binom{3x-1}{3}$		
$+\bigcap$ of 3			$+8\binom{4x-1}{3}$	$+24\binom{3x-1}{3}$	$+24\binom{2x-1}{3}$	
$-\bigcap$ of 4				$-6\binom{3x-1}{3}$	$-50\binom{2x-1}{3}$	$-14\binom{x-1}{3}$
$+\bigcap$ of 5					$+24\binom{2x-1}{3}$	$+32\binom{x-1}{3}$
$-\bigcap$ of 6					$-4\binom{2x-1}{3}$	$-24\binom{x-1}{3}$
$+\bigcap$ of 7						$+8\binom{x-1}{3}$
$-\bigcap$ of 8						$-\binom{x-1}{3}$
$\rho\mathrm{b}(30x+1,5)=$	$8\binom{8x-1}{3}$	$-12\binom{6x-1}{3}$	$+4\binom{4x-1}{3}$	$+6\binom{3x-1}{3}$	$-6\binom{2x-1}{3}$	$+\binom{x-1}{3}$

The table now accounts for the duplicate permissible patterns generated by all combinations of possible location sequences. Summing each column of binomials produces a combinatorial equation for the exact value of $\rho\mathrm{b}(30x+1,5)$. Evaluating this combinatorial equation for $x = 1$ correlates with the value provided in Table 3 for $w = 31$ and $k = 5$.

$$\begin{aligned}\rho\mathrm{b}(31,5) &= 8\binom{8-1}{3} - 12\binom{6-1}{3} + 4\binom{4-1}{3} + 6\binom{3-1}{3} - 6\binom{2-1}{3} + \binom{1-1}{3} \\ &= 8\cdot 35 - 12\cdot 10 + 4\cdot 1 + 6\cdot 0 - 6\cdot 0 + 0 \\ &= 164\end{aligned}$$

Performing this procedure of tallying the common possible locations in the possible location sequence combinations of $\rho\mathrm{b}(x\mathbb{P}_k + 1, k)$ for $k = 2$ through 10 generates the following combinatorial equations.

$$\rho\mathrm{b}(2x+1,2) = \binom{x-1}{0} = 1$$

$$\rho\mathrm{b}(6x+1,3) = 2\binom{2x-1}{1} - \binom{x-1}{1}$$

$$\rho\mathrm{b}(6x+1,4) = 2\binom{2x-1}{2} - \binom{x-1}{2}$$

$$\rho\mathrm{b}(30x+1,5) = 8\binom{8x-1}{3} - 12\binom{6x-1}{3} + 4\binom{4x-1}{3} + 6\binom{3x-1}{3} - 6\binom{2x-1}{3} + \binom{x-1}{3}$$

$$\rho\mathrm{b}(30x+1,6) = 8\binom{8x-1}{4} - 12\binom{6x-1}{4} + 4\binom{4x-1}{4} + 6\binom{3x-1}{4} - 6\binom{2x-1}{4} + \binom{x-1}{4}$$

$$\begin{aligned}\rho\mathrm{b}(210x+1,7) = {} & 48\binom{48x-1}{5} - 120\binom{40x-1}{5} - 72\binom{36x-1}{5} + 160\binom{32x-1}{5} + 180\binom{30x-1}{5} \\ & -336\binom{24x-1}{5} - 60\binom{20x-1}{5} + 216\binom{18x-1}{5} + 128\binom{16x-1}{5} - 90\binom{15x-1}{5} \\ & -48\binom{12x-1}{5} + 90\binom{10x-1}{5} - 90\binom{9x-1}{5} - 104\binom{8x-1}{5} + 144\binom{6x-1}{5} \\ & -15\binom{5x-1}{5} - 20\binom{4x-1}{5} - 21\binom{3x-1}{5} + 12\binom{2x-1}{5} - \binom{x-1}{5}\end{aligned}$$

$$\begin{aligned}\rho\mathrm{b}(210x+1,8) = {} & 48\binom{48x-1}{6} - 120\binom{40x-1}{6} - 72\binom{36x-1}{6} + 160\binom{32x-1}{6} + 180\binom{30x-1}{6} \\ & -336\binom{24x-1}{6} - 60\binom{20x-1}{6} + 216\binom{18x-1}{6} + 128\binom{16x-1}{6} - 90\binom{15x-1}{6} \\ & -48\binom{12x-1}{6} + 90\binom{10x-1}{6} - 90\binom{9x-1}{6} - 104\binom{8x-1}{6} + 144\binom{6x-1}{6} \\ & -15\binom{5x-1}{6} - 20\binom{4x-1}{6} - 21\binom{3x-1}{6} + 12\binom{2x-1}{6} - \binom{x-1}{6}\end{aligned}$$

$$\begin{aligned}\rho\mathrm{b}(210x+1,9) = {} & 48\binom{48x-1}{7} - 120\binom{40x-1}{7} - 72\binom{36x-1}{7} + 160\binom{32x-1}{7} + 180\binom{30x-1}{7} \\ & -336\binom{24x-1}{7} - 60\binom{20x-1}{7} + 216\binom{18x-1}{7} + 128\binom{16x-1}{7} - 90\binom{15x-1}{7} \\ & -48\binom{12x-1}{7} + 90\binom{10x-1}{7} - 90\binom{9x-1}{7} - 104\binom{8x-1}{7} + 144\binom{6x-1}{7} \\ & -15\binom{5x-1}{7} - 20\binom{4x-1}{7} - 21\binom{3x-1}{7} + 12\binom{2x-1}{7} - \binom{x-1}{7}\end{aligned}$$

$$\begin{aligned}\rho\mathrm{b}(210x+1,10) = {} & 48\binom{48x-1}{8} - 120\binom{40x-1}{8} - 72\binom{36x-1}{8} + 160\binom{32x-1}{8} + 180\binom{30x-1}{8} \\ & -336\binom{24x-1}{8} - 60\binom{20x-1}{8} + 216\binom{18x-1}{8} + 128\binom{16x-1}{8} - 90\binom{15x-1}{8} \\ & -48\binom{12x-1}{8} + 90\binom{10x-1}{8} - 90\binom{9x-1}{8} - 104\binom{8x-1}{8} + 144\binom{6x-1}{8} \\ & -15\binom{5x-1}{8} - 20\binom{4x-1}{8} - 21\binom{3x-1}{8} + 12\binom{2x-1}{8} - \binom{x-1}{8}\end{aligned}$$

The binomials in these combinatorial equations are all based on sets of elements that are linear functions of x, therefore the combinatorial equations can be converted into polynomial equations of x. The order of the polynomial is based on the subgroup size which is $k-2$ for each binomial. The following are polynomial equations equivalent to the combinatorial equations of $\rho\mathrm{b}(x\mathbb{P}_k+1,k)$ for $k=2$ through 10.

$$\begin{aligned}
\rho\mathrm{b}(2x+1,2) &= 1 \\
\rho\mathrm{b}(6x+1,3) &= \tfrac{3}{1}x-\tfrac{1}{1} \\
\rho\mathrm{b}(6x+1,4) &= \tfrac{7}{2}x^2-\tfrac{9}{2}x+\tfrac{2}{2} \\
\rho\mathrm{b}(30x+1,5) &= \tfrac{1875}{6}x^3-\tfrac{1050}{6}x^2+\tfrac{165}{6}x-\tfrac{6}{6} \\
\rho\mathrm{b}(30x+1,6) &= \tfrac{18631}{24}x^4-\tfrac{18750}{24}x^3+\tfrac{6125}{24}x^2-\tfrac{750}{24}x+\tfrac{24}{24} \\
\rho\mathrm{b}(210x+1,7) &= \tfrac{2927695365}{120}x^5-\tfrac{670995465}{120}x^4+\tfrac{54665625}{120}x^3-\tfrac{1929375}{120}x^2 \\
&\quad +\tfrac{28770}{120}x-\tfrac{120}{120} \\
\rho\mathrm{b}(210x+1,8) &= \tfrac{182135041495}{720}x^6-\tfrac{61481602665}{720}x^5+\tfrac{7828280425}{720}x^4-\tfrac{472696875}{720}x^3 \\
&\quad +\tfrac{13925800}{720}x^2-\tfrac{185220}{720}x+\tfrac{720}{720} \\
\rho\mathrm{b}(210x+1,9) &= \tfrac{10842356545125}{5040}x^7-\tfrac{5099781161860}{5040}x^6+\tfrac{942717907530}{5040}x^5-\tfrac{87676740760}{5040}x^4 \\
&\quad +\tfrac{4353313125}{5040}x^3-\tfrac{112606900}{5040}x^2+\tfrac{1372140}{5040}x-\tfrac{5040}{5040} \\
\rho\mathrm{b}(210x+1,10) &= \tfrac{621234485684071}{40320}x^8-\tfrac{390324835624500}{40320}x^7+\tfrac{99445732656270}{40320}x^6 \\
&\quad -\tfrac{13280026175640}{40320}x^5+\tfrac{1004211812919}{40320}x^4-\tfrac{43272022500}{40320}x^3 \\
&\quad +\tfrac{1012913300}{40320}x^2-\tfrac{11506320}{40320}x+\tfrac{40320}{40320}
\end{aligned}$$

Even though some of these polynomial equations can be simplified, such as the polynomial equation for $\rho\mathrm{b}(6x+1,4)$, the polynomials are left in raw form so the coefficients and their structure can be investigated.

$$\rho\mathrm{b}(6x+1,4) \quad = \quad \tfrac{7}{2}x^2-\tfrac{9}{2}x+\tfrac{2}{2} \quad = \quad \tfrac{1}{2}(7x-2)(x-1)$$

The combinatorial equations are converted to polynomial equations by expanding the binomials and canceling common terms.

$$\text{e.g.} \quad \binom{8x-1}{3} = \frac{(8x-1)!}{(8x-4)!\,3!} = \frac{(8x-1)(8x-2)(8x-3)}{3!}$$

The numerator in each expansion is converted into a falling factorial. The falling factorial is denoted as $(x)^{\underline{n}}$.

$$(x)^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$$

The permissible pattern functions are now converted to a sum of falling factorial multiples divided by a factorial.

$$\begin{aligned}\rho\text{b}(30x+1,5) = \frac{1}{3!}\Big(&8(8x-1)^{\underline{3}} - 12(6x-1)^{\underline{3}} + 4(4x-1)^{\underline{3}} \\ &+ 6(3x-1)^{\underline{3}} - 6(2x-1)^{\underline{3}} + (x-1)^{\underline{3}}\Big)\end{aligned}$$

The presence of the falling factorials invoke using an identity for signed Stirling numbers of the first kind. The signed Stirling numbers of the first kind are represented as $s(n,i)$.

$$(x)^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i$$

Substituting the Stirling number identity for each falling factorial and then regrouping the terms converts the permissible pattern counting function into a summation.

$$\begin{aligned}\rho\text{b}(30x+1,5) = \frac{1}{3!}\sum_{i=0}^{3} s(3,i)\Big(&8(8x-1)^i - 12(6x-1)^i + 4(4x-1)^i \\ &+ 6(3x-1)^i - 6(2x-1)^i + (x-1)^i\Big)\end{aligned}$$

The signed Stirling numbers of the first kind are replaced with unsigned Stirling numbers of the first kind with the appropriate sign, the powers are expanded, and then the terms are again regrouped. Unsigned Stirling numbers of the first kind represent the number of ways to permute a set of n elements into i cycles. The unsigned Stirling numbers of the first kind are denoted as $\left[{n \atop i}\right]$ where $\left[{n \atop i}\right] = |s(n,i)|$.

$$\begin{aligned}\rho\text{b}(30x+1,5) = \frac{1}{3!}\Big(&1875x^3\sum_{i=3}^{3}\begin{bmatrix}3\\i\end{bmatrix}\binom{i}{3} - 175x^3\sum_{i=2}^{3}\begin{bmatrix}3\\i\end{bmatrix}\binom{i}{2} \\ &+ 15x^3\sum_{i=1}^{3}\begin{bmatrix}3\\i\end{bmatrix}\binom{i}{1} - \sum_{i=0}^{3}\begin{bmatrix}3\\i\end{bmatrix}\binom{i}{0}\Big)\end{aligned}$$

The summation of unsigned Stirling numbers of the first kind paired with a binomial allows another identity to be used.

$$\sum_{i=a}^{n}\begin{bmatrix}n\\i\end{bmatrix}\binom{i}{a} = \begin{bmatrix}n+1\\i+1\end{bmatrix}$$

The identity is applied to the summations.

$$\rho\text{b}(30x+1,5) = \tfrac{1}{3!}\Big(1875\left[{4 \atop 4}\right]x^3 - 175\left[{4 \atop 3}\right]x^2 + 15\left[{4 \atop 2}\right]x - \left[{4 \atop 1}\right]\Big)$$

Finally, the unsigned Stirling numbers of the first kind are replaced with the signed Stirling numbers of the first kind to create a polynomial in terms of x.

$$\rho\text{b}(30x+1,5) = \frac{1}{3!}\left(1875\,s(4,4)x^3 + 175\,s(4,3)x^2 + 15\,s(4,2)x + s(4,1)\right)$$

A brief table of signed Stirling numbers of the first kind is shown below for reference.

Signed Stirling Numbers of the First Kind $s(m, n)$

m	value of n									
	9	8	7	6	5	4	3	2	1	0
0										1
1									1	0
2								1	-1	0
3							1	-3	2	0
4						1	-6	11	-6	0
5					1	-10	35	-50	24	0
6				1	-15	85	-225	274	-120	0
7			1	-21	175	-735	1624	-1764	720	0
8		1	-28	322	-1960	6769	-13132	13068	-5040	0
9	1	-36	546	-4536	22449	-67284	118124	-109584	40320	0

The polynomial equation representing $\rho\text{b}(x\mathbb{P}_k + 1, k)$ consists of $k - 1$ terms and the coefficients of the terms are multiples of $s(k-1, i)$ for $i = 1$ to $k - 1$. The polynomial equations of $\rho\text{b}(x\mathbb{P}_k + 1, k)$ for $k = 2$ through 10 are rewritten with the factors that are Stirling numbers of the first kind underlined. Also, the common factorial has been extracted.

$$\rho\text{b}(2x+1,2) = \tfrac{1}{0!}\big(\underline{1}\big) = 1$$

$$\rho\text{b}(6x+1,3) = \tfrac{1}{1!}\big(3\cdot\underline{1}x - \underline{1}\big)$$

$$\rho\text{b}(6x+1,4) = \tfrac{1}{2!}\big(7\cdot\underline{1}x^2 - 3\cdot\underline{3}x + \underline{2}\big)$$

$$\rho\text{b}(30x+1,5) = \tfrac{1}{3!}\big(125\cdot 15\cdot\underline{1}x^3 - 25\cdot 7\cdot\underline{6}x^2 + 5\cdot 3\cdot\underline{11}x - \underline{6}\big)$$

$$\rho\text{b}(30x+1,6) = \tfrac{1}{4!}\big(601\cdot 31\cdot\underline{1}x^4 - 125\cdot 15\cdot\underline{10}x^3 + 25\cdot 7\cdot\underline{35}x^2 - 5\cdot 3\cdot\underline{50}x + \underline{24}\big)$$

$$\rho\text{b}(210x+1,7) = \tfrac{1}{5!}\big(16807\cdot 2765\cdot 63\cdot\underline{1}x^5 - 2401\cdot 601\cdot 31\cdot\underline{15}x^4 + 343\cdot 125\cdot 15\cdot\underline{85}x^3 - 49\cdot 25\cdot 7\cdot\underline{225}x^2 + 7\cdot 5\cdot 3\cdot\underline{274}x - \underline{120}\big)$$

$$\rho\text{b}(210x+1,8) = \tfrac{1}{6!}\big(116929\cdot 12265\cdot 127\cdot\underline{1}x^6 - 16807\cdot 2765\cdot 63\cdot\underline{21}x^5 + 2401\cdot 601\cdot 31\cdot\underline{175}x^4 - 343\cdot 125\cdot 15\cdot\underline{735}x^3 + 49\cdot 25\cdot 7\cdot\underline{1624}x^2 - 7\cdot 5\cdot 3\cdot\underline{1764}x + \underline{720}\big)$$

$$\rho\text{b}(210x+1,9) = \tfrac{1}{7!}\big(803383\cdot 52925\cdot 255\cdot\underline{1}x^7 - 116929\cdot 12265\cdot 127\cdot\underline{28}x^6 + 16807\cdot 2765\cdot 63\cdot\underline{322}x^5 - 2401\cdot 601\cdot 31\cdot\underline{1960}x^4 + 343\cdot 125\cdot 15\cdot\underline{6769}x^3 - 49\cdot 25\cdot 7\cdot\underline{13132}x^2 + 7\cdot 5\cdot 3\cdot\underline{13068}x - \underline{5040}\big)$$

$$\rho\text{b}(210x+1,10) = \tfrac{1}{8!}\big(5432161\cdot 223801\cdot 511\cdot\underline{1}x^8 - 803383\cdot 52925\cdot 255\cdot\underline{36}x^7 + 116929\cdot 12265\cdot 127\cdot\underline{546}x^6 - 16807\cdot 2765\cdot 63\cdot\underline{4536}x^5 + 2401\cdot 601\cdot 31\cdot\underline{22449}x^4 - 343\cdot 125\cdot 15\cdot\underline{67284}x^3 + 49\cdot 25\cdot 7\cdot\underline{118124}x^2 - 7\cdot 5\cdot 3\cdot\underline{109584}x + \underline{40320}\big)$$

The structure of the factors that remain can be described using the analogy of counting the numbers that do not contain all digits 1 through $a-1$ simultaneously when the numbers 0 through a^n-1 are written in base a. The arrays below display the analogy for $a=3$ with $n=0$, 1, 2, and 3. The 'lined-out' numbers are not counted because the digits 1 and 2 are both present. The count is expressed as the two variable function $f(n,a)$.

$f(0,3)=1$	$f(1,3)=3$	$f(2,3)=7$	$f(3,3)=15$
			000 100 200
			001 101 ~~201~~
			002 ~~102~~ 202
		00 10 20	010 110 ~~210~~
0	0 1 2	01 11 ~~21~~	011 111 ~~211~~
		02 ~~12~~ 22	~~012~~ ~~112~~ ~~212~~
			020 ~~120~~ 220
			~~021~~ ~~121~~ ~~221~~
			022 ~~122~~ 222

The variable a represents the base to be used for counting. When counting in base a there are a digits available, namely the digits 0, 1, 2, ..., $a-1$. Valid bases for counting must have at least 2 different digits, meaning the variable a must be greater than or equal to 2. The variable n represents the duration of the counting where the numbers to be counted start a 0 and continue through a^n-1.

The value of $a^n-1=0$ for all a when $n=0$ and the largest number to be counted is 0. The single digit 0 is the only number counted, therefore when $n=0$ the quantity of numbers counted equals 1.

$$f(0,a)=1 \quad \text{for all} \quad a\geq 2 \tag{16}$$

The definition of the function is counting the numbers that do not contain all digits 1 through $a-1$. The digits 0 and 1 are used for counting in base 2. Every number written in base 2, except 0, contains at least one 1. The only number counted is 0, therefore when $a=2$ the quantity of numbers counted equals 1.

$$f(n,2)=1 \quad \text{for all} \quad n\geq 0$$

Numbers written in base a that are less than a^n have a maximum of n digits. When $n=a-2$ there is a maximum of $a-2$ digits contained in the numbers being counted but there are $a-1$ different digits from 1, 2, ..., $a-1$. Numbers less than a^{a-2} cannot simultaneously contain all $a-1$ digits, therefore when $n\leq a-2$ the quantity of numbers counted is a^n.

$$f(n,a)=a^n \quad \text{for all} \quad n\leq a-2$$

When $n=a-1$ there is a maximum of $a-1$ digits contained in the numbers being counted and there is also $a-1$ different digits from 1, 2, ..., $a-1$. Numbers containing $a-1$ digits can simultaneously contain one each of these $a-1$ different digits. There are $(a-1)!$ permutations of the $a-1$ different digits and each permutation represents a number that is not counted, therefore when $n = a - 1$ the quantity of numbers counted is a^{a-1} less the $(a-1)!$ permutations of the $a-1$ different digits.

$$f(a-1, a) = a^{a-1} - (a-1)!$$

The quantity of numbers for $f(n, a)$ can be determined by arranging the numbers 0 through $a^n - 1$ into columns of $a^{n-1} - 1$ consecutive numbers. The quantity of numbers in the first column that match the criteria for the analogy is $f(n-1, a)$ since the first digit is a zero. Also, in each of the remaining $a-1$ columns the quantity of numbers that match the criteria is $f(n-1, a)$ less a quantity $b_{n,a}$ since the first digit is not a zero.

$$\begin{aligned} f(n, a) &= f(n-1, a) + (a-1)\big(f(n-1, a) - b_{n,a}\big) \\ &= a\, f(n-1, a) - (a-1)\, b_{n,a} \end{aligned}$$

Cascading this equation permits $f(n, a)$ to be expressed in terms of previously calculated values of $f(n-i, a)$.

$$\begin{aligned} f(m+1, a) &= a\, f(m, a) - (a-1)\, b_{m+1,a} \\ f(m+2, a) &= a^2\, f(m, a) - (a-1)\big(a\, b_{m+1,a} + b_{m+2,a}\big) \\ f(m+3, a) &= a^3\, f(m, a) - (a-1)\big(a^2\, b_{m+1,a} + a\, b_{m+2,a} + b_{m+3,a}\big) \\ &\;\;\vdots \\ f(m+n, a) &= a^n\, f(m, a) - (a-1) \sum_{i=1}^{n} a^{n-i}\, b_{m+i,a} \end{aligned}$$

Setting $m = 0$ allows $f(m, a)$ to be canceled due to equation (16), thereby producing a summation of the $b_{i+1,a}$ quantities matched with powers of a.

$$f(n, a) = a^n - (a-1) \sum_{i=0}^{n-1} a^{n-i-1}\, b_{i+1,a}$$

Returning to the analogy, the value of $b_{n,a}$ is the quantity of numbers that are not counted due to the additional non-zero digit in the first column. The value $b_{n,a}$ counts the $(a-2)!$ permutations of $a-1$ nonempty subsets of the n digits. Stirling numbers of the second kind represented as $\left\{ {n \atop a-1} \right\}$ count the number of ways to partition a set of n elements into $a-1$ nonempty subsets. Using this notation for Stirling numbers of the second kind the value of $b_{n,a}$ is expressed by the following equation.

$$b_{n,a} = (a-2)! \left\{ {n \atop a-1} \right\}$$

Substituting the equivalent product for each $b_{i,a}$ produces a summation of Stirling numbers of the second kind matched with powers of a.

$$f(n, a) = a^n - (a-1)! \sum_{i=1}^{n} a^{n-i} \left\{ {i \atop a-1} \right\}$$

Using an identity of Stirling numbers of the second kind, a summation of powers multiplied by Stirling numbers of the second kind creates a single Stirling number of the second kind.

$$\left\{ {n+1 \atop a} \right\} = \sum_{i=0}^{n} a^{n-i} \left\{ {i \atop a-1} \right\}$$

Applying this identity to the equation produces a factorial times a single Stirling number of the second kind.

$$f(n,a) = a^n - (a-1)! \left\{ {n+1 \atop a} \right\}$$

The Stirling number of the second kind recurrence identity is used to create a sum of two Stirling numbers of the second kind.

$$\left\{ {n+1 \atop a} \right\} = a \left\{ {n \atop a} \right\} + \left\{ {n \atop a-1} \right\}$$

When this identity is applied to the equation two Stirling numbers of the second kind are produced. Each Stirling number of the second kind is now matched with a factorial of the subset size.

$$f(n,a) = a^n - a! \left\{ {n \atop a} \right\} - (a-1)! \left\{ {n \atop a-1} \right\}$$

A Stirling number of the second kind multiplied by a factorial of the subset size invokes the usage of another identity.

$$a! \left\{ {n \atop a} \right\} = \sum_{i=0}^{a} -1^{a-i} \binom{a}{i} i^n$$

Two summations of an alternating sign binomial times a power of the summation index are produced by applying this identity to the equations products of factorials and Stirling numbers of the second kind.

$$f(n,a) = a^n - \sum_{i=0}^{a} -1^{a-i} \binom{a}{i} i^n - \sum_{i=0}^{a-1} -1^{a-1-i} \binom{a-1}{i} i^n$$

The ath term is extracted from the first summation and canceled when subtracted from the existing a^n value. The remaining $a-1$ terms are paired with the $a-1$ terms of the second summation leaving a single summation of alternating sign differences of binomials times a power of the summation index.

$$f(n,a) = \sum_{i=0}^{a-1} -1^{a-1-i} \left(\binom{a}{i} - \binom{a-1}{i} \right) i^n$$

The binomial recurrence identity is used to create a single binomial.

$$\binom{a-1}{i-1} = \binom{a}{i} - \binom{a-1}{i}$$

A simple summation of alternating sign binomials times a power of the summation index is produced by applying this identity to the equation.

$$f(n,a) = \sum_{i=1}^{a-1} -1^{a-1-i} \binom{a-1}{i-1}\, i^{\,n}$$

One last simplification is made by expanding the summation and then summing in reverse order.

$$f(n,a) = \sum_{i=1}^{a-1} -1^{i-1} \binom{a-1}{i}\, (a-i)^n$$

The function $f(n,a)$ described above is used to determine factors of coefficients in the polynomial equations for $\rho\mathrm{b}(x\mathbb{P}_k + 1, k)$. The function $f(n,a)$ is formally denoted E_a^n. Calculated values of the function E_a^n are given in Table 4.

$$\mathrm{E}_a^n = \sum_{i=1}^{a-1} -1^{i-1} \binom{a-1}{i} (a-i)^n \tag{17}$$

The analogy given for the function E_a^n is related to admissible prime tuples when the digits in the analogy represent the residue classes of specific primes. The digit 0 in the analogy represents an unused residue class for a prime of magnitude a, while the remaining digits represent filled residue classes. When the digits 1 through $a-1$ are all represented causing the number to not be counted, the corresponding tuple is not admissible due to all residue classes being used. The coefficients of the $\rho\mathrm{b}(x\mathbb{P}_k + 1, k)$ polynomial equations have factors related to each prime less than or equal to k. Each factor is the value of E_a^n with a being the prime involved and n representing the exponent of the coefficients term.

All the components required to create a closed form equation for $\rho\mathrm{b}(x\mathbb{P}_k + 1, k)$ are available leaving only the construction of the equation. First, the equation is a sum of $k-1$ terms, each being a coefficient times a power of x divided by $(k-2)!$. Also, each coefficient is a multiple of a signed Stirling number of the first kind based on k and the term exponent i.

$$\rho\mathrm{b}(x\mathbb{P}_k + 1, k) = \frac{1}{(k-2)!} \sum_{i=0}^{k-2} C_i\, s(k-1, i+1)\, x^i$$

Finally, the remaining portion of the coefficient is a product of the E_a^n values for each prime less than or equal to k. The value of n is the term exponent i and the value of a is the prime p_j. Here $\pi(k)$ is the prime counting function representing the number of primes less than or equal to k.

$$C_i = \prod_{j=1}^{\pi(k)} \mathrm{E}_{p_j}^i$$

The signed Stirling numbers account for the alternating signs of the terms. The closed form equation for $\rho\mathrm{b}(x\mathbb{P}_k + 1, k)$ is now complete and provides the count of permissible patterns with a density of k when the width is a multiple of the primorial of k plus one.

$$\rho\mathrm{b}(x\mathbb{P}_k + 1, k) = \frac{1}{(k-2)!} \sum_{i=0}^{k-2} \left(s(k-1, i+1)\, x^i \prod_{j=1}^{\pi(k)} \mathrm{E}_{p_j}^i \right) \tag{18}$$

$\pi()$ is the prime counting function
and p_j is the j th prime number

$\rho\mathrm{b}(x\mathbb{P}_k, k) = 0$

Develop closed form for $\rho\mathrm{f}(x\mathbb{P}_k + 1, k)$

$$\begin{aligned}
\rho\mathrm{f}(2x+1, 2) &= \tbinom{x}{1} \\
\rho\mathrm{f}(6x+1, 3) &= 2\tbinom{2x}{2} - \tbinom{x}{2} \\
\rho\mathrm{f}(30x+1, 5) &= 8\tbinom{8x}{4} - 12\tbinom{6x}{4} + 4\tbinom{4x}{4} + 6\tbinom{3x}{4} - 6\tbinom{2x}{4} + \tbinom{x}{4}
\end{aligned}$$

$$\begin{aligned}
\rho\mathrm{b}(2x+1, 2) &= \tfrac{1}{1!}\left(\underline{1}x \right) \\
\rho\mathrm{b}(6x+1, 3) &= \tfrac{1}{2!}\left(7\cdot\underline{1}x^2 - 3\cdot\underline{1}x \right) \\
\rho\mathrm{b}(30x+1, 5) &= \tfrac{1}{4!}\left(601\cdot 31\cdot\underline{1}x^4 - 125\cdot 15\cdot\underline{6}x^3 + 25\cdot 7\cdot\underline{11}x^2 - 5\cdot 3\cdot\underline{6}x \right)
\end{aligned}$$

$$\rho\mathrm{f}(x\mathbb{P}_k + 1, k) = \frac{1}{(k-1)!} \sum_{i=1}^{k-1} \left(s(k-1, i)\, x^i \prod_{j=1}^{\pi(k)} \mathrm{E}_{p_j}^i \right) \tag{19}$$

$\rho\mathrm{f}(x\mathbb{P}_k, k) = \rho\mathrm{f}(x\mathbb{P}_k + 1, k) - \rho\mathrm{b}(x\mathbb{P}_k + 1, k)$
$\rho\mathrm{f}(x\mathbb{P}_k - 1, k) = \rho\mathrm{f}(x\mathbb{P}_k, k)$

Develop closed form for $\rho(x\mathbb{P}_k + 1, k)$

$\rho(x\mathbb{P}_k, k) = \rho(x\mathbb{P}_k + 1, k) - \rho\mathrm{f}(x\mathbb{P}_k + 1, k)$
$\rho(x\mathbb{P}_k - 1, k) = \rho(x\mathbb{P}_k, k) - \rho\mathrm{f}(x\mathbb{P}_k, k)$
$\rho(x\mathbb{P}_k - 2, k) = \rho(x\mathbb{P}_k - 1, k) - \rho\mathrm{f}(x\mathbb{P}_k - 1, k)$

Width of pattern for first occurrence of any density ...

Reference Table 5 (verified using exhaustive search methods)

Identify 3159 as first counter-example to Hardy-Littlewood 2nd conjecture

Identify 5943 as last occurrence

5943 needs final verification

Violation of 'large sieve' ...

Results applied to Hardy-Littlewood 'k-tuples' conjecture

Consequences of results

add Hans Riesel and his book

add Vern Huber

PARI as a valuable tool

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Table 1. Values of $\rho\rho(w)$ and $\rho(w,k)$

w	$\rho\rho(w)$	$\rho(w,k)$													
		0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	2	1	1												
2	3	1	2												
3	5	1	3	1											
4	7	1	4	2											
5	10	1	5	4											
6	13	1	6	6											
7	19	1	7	9	2										
8	25	1	8	12	4										
9	35	1	9	16	8	1									
10	45	1	10	20	12	2									
11	59	1	11	25	18	4									
12	73	1	12	30	24	6									
13	101	1	13	36	35	14	2								
14	129	1	14	42	46	22	4								
15	170	1	15	49	61	36	8								
16	211	1	16	56	76	50	12								
17	268	1	17	64	95	70	20	1							
18	325	1	18	72	114	90	28	2							
19	430	1	19	81	141	129	52	7							
20	535	1	20	90	168	168	76	12							
21	695	1	21	100	201	222	120	28	2						
22	855	1	22	110	234	276	164	44	4						
23	1065	1	23	121	273	345	226	70	6						
24	1275	1	24	132	312	414	288	96	8						
25	1658	1	25	144	362	522	412	168	24						
26	2041	1	26	156	412	630	536	240	40						
27	2572	1	27	169	470	766	708	354	74	3					
28	3103	1	28	182	528	902	880	468	108	6					
29	3781	1	29	196	594	1066	1100	624	160	11					
30	4459	1	30	210	660	1230	1320	780	212	16					
31	5802	1	31	225	740	1460	1704	1156	414	67	4				
32	7145	1	32	240	820	1690	2088	1532	616	118	8				
33	9068	1	33	256	910	1965	2584	2074	966	245	32	2			
34	10991	1	34	272	1000	2240	3080	2616	1316	372	56	4			
35	13473	1	35	289	1100	2560	3688	3324	1810	565	94	7			
36	15955	1	36	306	1200	2880	4296	4032	2304	758	132	10			
37	20357	1	37	324	1317	3300	5201	5253	3328	1275	284	35	2		
38	24759	1	38	342	1434	3720	6106	6474	4352	1792	436	60	4		
39	30608	1	39	361	1563	4206	7213	8073	5800	2576	674	96	6		
40	36457	1	40	380	1692	4692	8320	9672	7248	3360	912	132	8		
41	44281	1	41	400	1833	5244	9647	11730	9280	4573	1320	200	12		
42	52105	1	42	420	1974	5796	10974	13788	11312	5786	1728	268	16		
43	66169	1	43	441	2135	6489	12819	16996	15010	8549	3010	612	62	2	
44	80233	1	44	462	2296	7182	14664	20204	18708	11312	4292	956	108	4	
45	98525	1	45	484	2471	7966	16837	24143	23468	15099	6224	1561	214	12	
46	116817	1	46	506	2646	8750	19010	28082	28228	18886	8156	2166	320	20	
47	140798	1	47	529	2835	9625	21529	32869	34370	25154	11096	3186	520	37	
48	164779	1	48	552	3024	10500	24048	37656	40512	29422	14036	4206	720	54	
49	204524	1	49	576	3236	11564	27374	44538	50204	38711	20012	6722	1380	151	6
50	244269	1	50	600	3448	12628	30700	51420	59896	48000	25988	9238	2040	248	12
51	301576														
52	358883														
53	430522														
54	502161														
55	620007														
56	737853														
57	894770														
58	1051687														
59	1243921														
60	1436155														
61	1800700														
62	2165245														

The $\rho\rho()$ column is A023192 in *The On-Line Encyclopedia of Integer Sequences*

TABLE 2. Values of $\rho\rho\mathrm{f}(w)$ and $\rho\mathrm{f}(w,k)$

w	$\rho\rho\mathrm{f}(w)$	$\rho\mathrm{f}(w,k)$ 1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1													
2	1	1													
3	2	1	1												
4	2	1	1												
5	3	1	2												
6	3	1	2												
7	6	1	3	2											
8	6	1	3	2											
9	10	1	4	4	1										
10	10	1	4	4	1										
11	14	1	5	6	2										
12	14	1	5	6	2										
13	28	1	6	11	8	2									
14	28	1	6	11	8	2									
15	41	1	7	15	14	4									
16	41	1	7	15	14	4									
17	57	1	8	19	20	8	1								
18	57	1	8	19	20	8	1								
19	105	1	9	27	39	24	5								
20	105	1	9	27	39	24	5								
21	160	1	10	33	54	44	16	2							
22	160	1	10	33	54	44	16	2							
23	210	1	11	39	69	62	26	2							
24	210	1	11	39	69	62	26	2							
25	383	1	12	50	108	124	72	16							
26	383	1	12	50	108	124	72	16							
27	531	1	13	58	136	172	114	34	3						
28	531	1	13	58	136	172	114	34	3						
29	678	1	14	66	164	220	156	52	5						
30	678	1	14	66	164	220	156	52	5						
31	1343	1	15	80	230	384	376	202	51	4					
32	1343	1	15	80	230	384	376	202	51	4					
33	1923	1	16	90	275	496	542	350	127	24	2				
34	1923	1	16	90	275	496	542	350	127	24	2				
35	2482	1	17	100	320	608	708	494	193	38	3				
36	2482	1	17	100	320	608	708	494	193	38	3				
37	4402	1	18	117	420	905	1221	1024	517	152	25	2			
38	4402	1	18	117	420	905	1221	1024	517	152	25	2			
39	5849	1	19	129	486	1107	1599	1448	784	238	36	2			
40	5849	1	19	129	486	1107	1599	1448	784	238	36	2			
41	7824	1	20	141	552	1327	2058	2032	1213	408	68	4			
42	7824	1	20	141	552	1327	2058	2032	1213	408	68	4			
43	14064	1	21	161	693	1845	3208	3698	2763	1282	344	46	2		
44	14064	1	21	161	693	1845	3208	3698	2763	1282	344	46	2		
45	18292	1	22	175	784	2173	3939	4760	3787	1932	605	106	8		
46	18292	1	22	175	784	2173	3939	4760	3787	1932	605	106	8		
47	23981	1	23	189	875	2519	4787	6142	5268	2940	1020	200	17		
48	23981	1	23	189	875	2519	4787	6142	5268	2940	1020	200	17		
49	39745	1	24	212	1064	3326	6882	9692	9289	5976	2516	660	97	6	
50	39745	1	24	212	1064	3326	6882	9692	9289	5976	2516	660	97	6	
51	57307	1	25	228	1184	3886	8558	12994	13659	9896	4891	1608	335	40	2
52	57307	1	25	228	1184	3886	8558	12994	13659	9896	4891	1608	335	40	2
53	71639	1	26	244	1304	4410	10036	15802	17284	13050	6699	2260	469	52	2
54	71639	1	26	244	1304	4410	10036	15802	17284	13050	6699	2260	469	52	2
55	117846	1	27	270	1548	5626	13794	23536	28186	23696	13895	5568	1457	226	16
56	117846	1	27	270	1548	5626	13794	23536	28186	23696	13895	5568	1457	226	16
57	156917														
58	156917														
59	192234														
60	192234														
61	364545														
62	363545														

Table 3. Values of $\rho\rho\text{b}(w)$ and $\rho\text{b}(w,k)$

w	$\rho\rho\text{b}(w)$	$\rho\text{b}(w,k)$														
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	1														
5	1	1														
7	3	1	2													
9	4	1	2	1												
11	4	1	2	1												
13	14	1	5	6	2											
15	13	1	4	6	2											
17	16	1	4	6	4	1										
19	48	1	8	19	16	4										
21	55	1	6	15	20	11	2									
23	50	1	6	15	18	10										
25	173	1	11	39	62	46	14									
27	148	1	8	28	48	42	18	3								
29	147	1	8	28	48	42	18	2								
31	665	1	14	66	164	220	150	46	4							
33	580	1	10	45	112	166	148	76	20	2						
35	559	1	10	45	112	166	144	66	14	1						
37	1920	1	17	100	297	513	530	324	114	22	2					
39	1447	1	12	66	202	378	424	267	86	11						
41	1975	1	12	66	220	459	584	429	170	32	2					
43	6240	1	20	141	518	1150	1666	1550	874	276	42	2				
45	4228	1	14	91	328	731	1062	1024	650	261	60	6				
47	5689	1	14	91	346	848	1382	1481	1008	415	94	9				
49	15764	1	23	189	807	2095	3550	4021	3036	1496	460	80	6			
51	17562	1	16	120	560	1676	3302	4370	3920	2375	948	238	34	2		
53	14332	1	16	120	524	1478	2808	3625	3154	1808	652	134	12			
55	46207	1	26	244	1216	3758	7734	10902	10646	7196	3308	988	174	14		
57	39071	1	18	153	752	2388	5256	8209	9126	7212	4000	1517	378	57	4	
59	35317	1	18	153	752	2388	5160	7772	8248	6174	3220	1137	258	34	2	
61	172311	1	29	306	1854	7130	18295	32362	40316	35826	22680	9998	2930	530	52	2

Note: $\rho\rho\text{b}(w) = 0$ and $\rho\text{b}(w,k) = 0$ for all even values of w
The $\rho\rho\text{b}()$ column is A023189 in *The On-Line Encyclopedia of Integer Sequences*

Table 4. Values of E_a^n function

a	value of n									
	0	1	2	3	4	5	6	7	8	9
2	2^0	1	1	1	1	1	1	1	1	1
3	3^0	3^1	7	15	31	63	127	255	511	1023
4	4^0	4^1	4^2	58	196	634	1996	6178	18916	57514
5	5^0	5^1	5^2	5^3	601	2765	12265	52925	223801	932525
6	6^0	6^1	6^2	6^3	6^4	7656	44136	248016	1362096	7338456
7	7^0	7^1	7^2	7^3	7^4	7^5	116929	803383	5432161	36120007
8	8^0	8^1	8^2	8^3	8^4	8^5	8^6	2092112	16777216	131889248
9	9^0	9^1	9^2	9^3	9^4	9^5	9^6	9^7	43046721	385968969
10	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	999637120
11	11^0	11^1	11^2	11^3	11^4	11^5	11^6	11^7	11^8	11^9

Table anti-diagonals are A158198 in *The On-Line Encyclopedia of Integer Sequences*

Table 5. Minimum width (W) for a given density (D)

D	W	D	W	D	W	D	W	D	W	D	W	D	W
1	1	51	253	101	573	151	909	201	1275	251	1645	301	2017
2	3	52	255	102	577	152	913	202	1281	252	1651	302	2023
3	7	53	265	103	579	153	927	203	1291	253	1657	303	2027
4	9	54	271	104	591	154	931	204	1303	254	1667	304	2035
5	13	55	273	105	601	155	935	205	1309	255	1673	305	2047
6	17	56	279	106	603	156	947	206	1317	256	1681	306	2051
7	21	57	283	107	607	157	953	207	1321	257	1687	307	2061
8	27	58	289	108	613	158	961	208	1329	258	1693	308	2065
9	31	59	301	109	617	159	971	209	1333	259	1701	309	2073
10	33	60	305	110	629	160	975	210	1339	260	1707	310	2077
11	37	61	311	111	635	161	987	211	1345	261	1717	311	2087
12	43	62	321	112	641	162	991	212	1351	262	1721	312	2101
13	49	63	325	113	647	163	999	213	1353	263	1729	313	2103
14	51	64	331	114	655	164	1003	214	1359	264	1737	314	2109
15	57	65	337	115	657	165	1013	215	1365	265	1747	315	2125
16	61	66	343	116	663	166	1023	216	1371	266	1753	316	2133
17	67	67	351	117	673	167	1027	217	1375	267	1761	317	2137
18	71	68	357	118	681	168	1033	218	1381	268	1765	318	2145
19	77	69	367	119	687	169	1037	219	1387	269	1773	319	2149
20	81	70	371	120	693	170	1045	220	1393	270	1783	320	2155
21	85	71	379	121	703	171	1051	221	1405	271	1791	321	2167
22	91	72	385	122	709	172	1059	222	1413	272	1797	322	2175
23	95	73	391	123	715	173	1067	223	1417	273	1803	323	2179
24	101	74	393	124	723	174	1071	224	1433	274	1813	324	2191
25	111	75	399	125	733	175	1075	225	1441	275	1823	325	2201
26	115	76	411	126	741	176	1083	226	1449	276	1827	326	2205
27	121	77	421	127	747	177	1087	227	1457	277	1837	327	2211
28	127	78	423	128	751	178	1105	228	1463	278	1843	328	2221
29	131	79	427	129	761	179	1111	229	1471	279	1849	329	2227
30	137	80	433	130	769	180	1121	230	1477	280	1855	330	2231
31	141	81	439	131	775	181	1125	231	1483	281	1863	331	2245
32	147	82	447	132	781	182	1131	232	1487	282	1871	332	2253
33	153	83	451	133	785	183	1143	233	1495	283	1877	333	2257
34	157	84	453	134	795	184	1147	234	1509	284	1883	334	2263
35	169	85	463	135	805	185	1151	235	1513	285	1891	335	2267
36	163	86	471	136	809	186	1163	236	1523	286	1895	336	2271
37	169	87	477	137	813	187	1169	237	1531	287	1901	337	2287
38	177	88	483	138	817	188	1177	238	1537	288	1915	338	2299
39	183	89	487	139	819	189	1183	239	1553	289	1921	339	2301
40	187	90	495	140	829	190	1189	240	1561	290	1927	340	2311
41	189	91	505	141	841	191	1195	241	1565	291	1933	341	2323
42	197	92	507	142	843	192	1201	242	1571	292	1941	342	2329
43	201	93	513	143	849	193	1205	243	1581	293	1945	343	2341
44	211	94	517	144	857	194	1211	244	1591	294	1963	344	2343
45	213	95	519	145	865	195	1219	245	1597	295	1967	345	2355
46	217	96	531	146	873	196	1231	246	1605	296	1981	346	2359
47	227	97	537	147	879	197	1239	247	1611	297	1987	347	2365
48	237	98	547	148	883	198	1259	248	1621	298	1993	348	2377
49	241	99	553	149	893	199	1263	249	1631	299	2001	349	2383
50	247	100	559	150	903	200	1267	250	1637	300	2011	350	2389

These values are A020497 in *The On-Line Encyclopedia of Integer Sequences*
Densities for all widths ≤ 2301 verified using exhaustive search methods

Paper in progress . . . June 3, 2009

misc text and equations not yet used

Binomial coefficients created from possible location counts for $\rho\mathrm{b}(x\mathbb{P}_5 + b, 5)$

b	binomial set size $8x+a$								$6x+a$						$4x+a$				$3x+a$			$2x+a$		x+a
	6	5	4	3	2	1	0	-1	4	3	2	1	0	-1	2	1	0	-1	1	0	-1	0	-1	-1
1								8						-12				4			6		-6	1
2																								
3							3						-3				1							
4																								
5							3						-3				1							
6																								
7						4	2					-2	-4				-1			3		-1		
8																								
9					1	2						-2	-1				1							
10																								
11					2	2					-1	-4	-1			2	2					-1		
12																								
13				2	4						-4	-2					-1		1	2		-1		
14																								
15				2	1						-1	-2					1							
16																								
17			1	2						-2	-1				1									
18																								
19			4	2						-2	-4				-1				2	1		-1		
20																								
21		2	2						-1	-4	-1				2	2						-1		
22																								
23		2	1						-1	-2					1									
24																								
25	2	4							-4	-2					-1				3			-1		
26																								
27	3								-3						1									
28																								
29	3								-3						1									
30																								

Polynomial equations for $\rho\mathrm{b}(x\mathbb{P}_5 + b, 5)$

$$\rho\mathrm{b}(30x+1,5) = \tfrac{1}{3!}\left(1875x^3 - 1050x^2 + 165x - 6\right)$$
$$\rho\mathrm{b}(30x+3,5) = \tfrac{1}{3!}\left(952x^3 - 300x^2 + 20x + 0\right)$$
$$\rho\mathrm{b}(30x+5,5) = \tfrac{1}{3!}\left(952x^3 - 300x^2 + 20x + 0\right)$$
$$\rho\mathrm{b}(30x+7,5) = \tfrac{1}{3!}\left(1785x^3 + 27x^2 - 30x + 0\right)$$
$$\rho\mathrm{b}(30x+9,5) = \tfrac{1}{3!}\left(952x^3 + 252x^2 + 8x + 0\right)$$
$$\rho\mathrm{b}(30x+11,5) = \tfrac{1}{3!}\left(1000x^3 + 300x^2 + 20x + 0\right)$$
$$\rho\mathrm{b}(30x+13,5) = \tfrac{1}{3!}\left(1785x^3 + 1110x^2 + 201x + 12\right)$$
$$\rho\mathrm{b}(30x+15,5) = \tfrac{1}{3!}\left(952x^3 + 804x^2 + 200x + 12\right)$$
$$\rho\mathrm{b}(30x+17,5) = \tfrac{1}{3!}\left(952x^3 + 852x^2 + 248x + 24\right)$$
$$\rho\mathrm{b}(30x+19,5) = \tfrac{1}{3!}\left(1785x^3 + 2145x^2 + 816x + 96\right)$$
$$\rho\mathrm{b}(30x+21,5) = \tfrac{1}{3!}\left(1000x^3 + 1500x^2 + 740x + 120\right)$$
$$\rho\mathrm{b}(30x+23,5) = \tfrac{1}{3!}\left(952x^3 + 1404x^2 + 680x + 108\right)$$
$$\rho\mathrm{b}(30x+25,5) = \tfrac{1}{3!}\left(1785x^3 + 3228x^2 + 1911x + 372\right)$$
$$\rho\mathrm{b}(30x+27,5) = \tfrac{1}{3!}\left(952x^3 + 1956x^2 + 1316x + 288\right)$$
$$\rho\mathrm{b}(30x+29,5) = \tfrac{1}{3!}\left(952x^3 + 1956x^2 + 1316x + 288\right)$$

Binomial coefficients created from possible location counts for $\rho\mathrm{f}(x\mathbb{P}_5 + b, 5)$

b	binomial set size $8x+a$								$6x+a$						$4x+a$				$3x+a$			$2x+a$		x+a
	7	6	5	4	3	2	1	0	5	4	3	2	1	0	3	2	1	0	2	1	0	1	0	0
1								8						-12				4			6		-6	1
2								8						-12				4			6		-6	1
3							3	5					-3	-9			1	3			6		-6	1
4							3	5					-3	-9			1	3			6		-6	1
5							6	2					-6	-6			2	2			6		-6	1
6							6	2					-6	-6			2	2			6		-6	1
7						4	4					-2	-8	-2			1	3		3	3	-1	-5	1
8						4	4					-2	-8	-2			1	3		3	3	-1	-5	1
9					1	5	2					-4	-7	-1			2	2		3	3	-1	-5	1
10					1	5	2					-4	-7	-1			2	2		3	3	-1	-5	1
11					3	5					-1	-7	-4			2	2			3	3	-2	-4	1
12					3	5					-1	-7	-4			2	2			3	3	-2	-4	1
13				2	5	1					-5	-5	-2			2	1	1	1	4	1	-3	-3	1
14				2	5	1					-5	-5	-2			2	1	1	1	4	1	-3	-3	1
15				4	4						-6	-6				2	2		1	4	1	-3	-3	1
16				4	4						-6	-6				2	2		1	4	1	-3	-3	1
17			1	5	2					-2	-5	-5			1	1	2		1	4	1	-3	-3	1
18			1	5	2					-2	-5	-5			1	1	2		1	4	1	-3	-3	1
19			5	3						-4	-7	-1				2	2		3	3		-4	-2	1
20			5	3						-4	-7	-1				2	2		3	3		-4	-2	1
21		2	5	1					-1	-7	-4				2	2			3	3		-5	-1	1
22		2	5	1					-1	-7	-4				2	2			3	3		-5	-1	1
23		4	4						-2	-8	-2				3	1			3	3		-5	-1	1
24		4	4						-2	-8	-2				3	1			3	3		-5	-1	1
25	2	6							-6	-6					2	2			6			-6		1
26	2	6							-6	-6					2	2			6			-6		1
27	5	3							-9	-3					3	1			6			-6		1
28	5	3							-9	-3					3	1			6			-6		1
29	8								-12						4				6			-6		1
30	8								-12						4				6			-6		1

Polynomial equations for $\rho\text{f}(x\mathbb{P}_5 + b, 5)$

$$\rho\text{f}(30x+1,5) = \tfrac{1}{4!}\left(18631x^4 - 11250x^3 + 1925x^2 - 90x + 0\right)$$
$$\rho\text{f}(30x+3,5) = \tfrac{1}{4!}\left(18631x^4 - 7442x^3 + 725x^2 - 10x + 0\right)$$
$$\rho\text{f}(30x+5,5) = \tfrac{1}{4!}\left(18631x^4 - 3634x^3 - 475x^2 + 70x + 0\right)$$
$$\rho\text{f}(30x+7,5) = \tfrac{1}{4!}\left(18631x^4 + 3506x^3 - 367x^2 - 50x + 0\right)$$
$$\rho\text{f}(30x+9,5) = \tfrac{1}{4!}\left(18631x^4 + 7314x^3 + 641x^2 - 18x + 0\right)$$
$$\rho\text{f}(30x+11,5) = \tfrac{1}{4!}\left(18631x^4 + 11314x^3 + 1841x^2 + 62x + 0\right)$$
$$\rho\text{f}(30x+13,5) = \tfrac{1}{4!}\left(18631x^4 + 18454x^3 + 6281x^2 + 866x + 48\right)$$
$$\rho\text{f}(30x+15,5) = \tfrac{1}{4!}\left(18631x^4 + 22262x^3 + 9497x^2 + 1666x + 96\right)$$
$$\rho\text{f}(30x+17,5) = \tfrac{1}{4!}\left(18631x^4 + 26070x^3 + 12905x^2 + 2658x + 192\right)$$
$$\rho\text{f}(30x+19,5) = \tfrac{1}{4!}\left(18631x^4 + 33210x^3 + 21485x^2 + 5922x + 576\right)$$
$$\rho\text{f}(30x+21,5) = \tfrac{1}{4!}\left(18631x^4 + 37210x^3 + 27485x^2 + 8882x + 1056\right)$$
$$\rho\text{f}(30x+23,5) = \tfrac{1}{4!}\left(18631x^4 + 41018x^3 + 33101x^2 + 11602x + 1488\right)$$
$$\rho\text{f}(30x+25,5) = \tfrac{1}{4!}\left(18631x^4 + 48158x^3 + 46013x^2 + 19246x + 2976\right)$$
$$\rho\text{f}(30x+27,5) = \tfrac{1}{4!}\left(18631x^4 + 51966x^3 + 53837x^2 + 24510x + 4128\right)$$
$$\rho\text{f}(30x+29,5) = \tfrac{1}{4!}\left(18631x^4 + 55774x^3 + 61661x^2 + 29774x + 5280\right)$$

Binomial coefficients created from possible location counts for $\rho(x\mathbb{P}_5 + b, 5)$

	binomial set size																													
b	$8x+a$									$6x+a$							$4x+a$					$3x+a$				$2x+a$			x+a	
	8	7	6	5	4	3	2	1	0	6	5	4	3	2	1	0	4	3	2	1	0	3	2	1	0	2	1	0	1	0
1								8	22						-12	-48				4	26			6	54		-6	-84	1	29
2								16	14						-24	-36				8	22			12	48		-12	-78	2	28
3							3	18	9					-3	-30	-27			1	10	19			18	42		-18	-72	3	27
4							6	20	4					-6	-36	-18			2	12	16			24	36		-24	-66	4	26
5							12	16	2					-12	-36	-12			4	12	14			30	30		-30	-60	5	25
6							18	12						-18	-36	-6			6	12	12			36	24		-36	-54	6	24
7						4	18	8					-2	-24	-30	-4			7	14	9		3	36	21	-1	-40	-49	7	23
8						8	18	4					-4	-30	-24	-2			8	16	6		6	36	18	-2	-44	-44	8	22
9					1	12	15	2					-8	-33	-18	-1			10	16	4		9	36	15	-3	-48	-39	9	21
10					2	16	12						-12	-36	-12				12	16	2		12	36	12	-4	-52	-34	10	20
11					5	18	7					-1	-18	-33	-8			2	12	14	2		15	36	9	-6	-54	-30	11	19
12					8	20	2					-2	-24	-30	-4			4	12	12	2		18	36	6	-8	-56	-26	12	18
13				2	11	16	1					-7	-24	-27	-2			6	11	12	1	1	21	33	5	-11	-56	-23	13	17
14				4	14	12						-12	-24	-24				8	10	12		2	24	30	4	-14	-56	-20	14	16
15				8	14	8						-18	-24	-18				10	10	10		3	27	27	3	-17	-56	-17	15	15
16				12	14	4						-24	-24	-12				12	10	8		4	30	24	2	-20	-56	-14	16	14
17			1	16	11	2					-2	-27	-24	-7			1	12	11	6		5	33	21	1	-23	-56	-11	17	13
18			2	20	8						-4	-30	-24	-2			2	12	12	4		6	36	18		-26	-56	-8	18	12
19			7	18	5						-8	-33	-18	-1			2	14	12	2		9	36	15		-30	-54	-6	19	11
20			12	16	2						-12	-36	-12				2	16	12			12	36	12		-34	-52	-4	20	10
21		2	15	12	1					-1	-18	-33	-8				4	16	10			15	36	9		-39	-48	-3	21	9
22		4	18	8						-2	-24	-30	-4				6	16	8			18	36	6		-44	-44	-2	22	8
23		8	18	4						-4	-30	-24	-2				9	14	7			21	36	3		-49	-40	-1	23	7
24		12	18							-6	-36	-18					12	12	6			24	36			-54	-36		24	6
25	2	16	12							-12	-36	-12					14	12	4			30	30			-60	-30		25	5
26	4	20	6							-18	-36	-6					16	12	2			36	24			-66	-24		26	4
27	9	18	3							-27	-30	-3					19	10	1			42	18			-72	-18		27	3
28	14	16								-36	-24						22	8				48	12			-78	-12		28	2
29	22	8								-48	-12						26	4				54	6			-84	-6		29	1
30	30									-60							30					60				-90			30	

Polynomial equations for $\rho(x\mathbb{P}_5 + b, 5)$

$$\rho(30x+0,5) = \tfrac{1}{5!}\left(558930x^5 - 562500x^4 + 183750x^3 - 22500x^2 + 720x + 0\right)$$
$$\rho(30x+1,5) = \tfrac{1}{5!}\left(558930x^5 - 469345x^4 + 127500x^3 - 12875x^2 + 270x + 0\right)$$
$$\rho(30x+2,5) = \tfrac{1}{5!}\left(558930x^5 - 376190x^4 + 71250x^3 - 3250x^2 - 180x + 0\right)$$
$$\rho(30x+3,5) = \tfrac{1}{5!}\left(558930x^5 - 283035x^4 + 34040x^3 + 375x^2 - 230x + 0\right)$$
$$\rho(30x+4,5) = \tfrac{1}{5!}\left(558930x^5 - 189880x^4 - 3170x^3 + 4000x^2 - 280x + 0\right)$$
$$\rho(30x+5,5) = \tfrac{1}{5!}\left(558930x^5 - 96725x^4 - 21340x^3 + 1625x^2 + 70x + 0\right)$$
$$\rho(30x+6,5) = \tfrac{1}{5!}\left(558930x^5 - 3570x^4 - 39510x^3 - 750x^2 + 420x + 0\right)$$
$$\rho(30x+7,5) = \tfrac{1}{5!}\left(558930x^5 + 89585x^4 - 21980x^3 - 2585x^2 + 170x + 0\right)$$
$$\rho(30x+8,5) = \tfrac{1}{5!}\left(558930x^5 + 182740x^4 - 4450x^3 - 4420x^2 - 80x + 0\right)$$
$$\rho(30x+9,5) = \tfrac{1}{5!}\left(558930x^5 + 275895x^4 + 32120x^3 - 1215x^2 - 170x + 0\right)$$
$$\rho(30x+10,5) = \tfrac{1}{5!}\left(558930x^5 + 369050x^4 + 68690x^3 + 1990x^2 - 260x + 0\right)$$
$$\rho(30x+11,5) = \tfrac{1}{5!}\left(558930x^5 + 462205x^4 + 125260x^3 + 11195x^2 + 50x + 0\right)$$
$$\rho(30x+12,5) = \tfrac{1}{5!}\left(558930x^5 + 555360x^4 + 181830x^3 + 20400x^2 + 360x + 0\right)$$
$$\rho(30x+13,5) = \tfrac{1}{5!}\left(558930x^5 + 648515x^4 + 274100x^3 + 51805x^2 + 4690x + 240\right)$$
$$\rho(30x+14,5) = \tfrac{1}{5!}\left(558930x^5 + 741670x^4 + 366370x^3 + 83210x^2 + 9020x + 480\right)$$
$$\rho(30x+15,5) = \tfrac{1}{5!}\left(558930x^5 + 834825x^4 + 477680x^3 + 130695x^2 + 17350x + 960\right)$$
$$\rho(30x+16,5) = \tfrac{1}{5!}\left(558930x^5 + 927980x^4 + 588990x^3 + 178180x^2 + 25680x + 1440\right)$$
$$\rho(30x+17,5) = \tfrac{1}{5!}\left(558930x^5 + 1021135x^4 + 719340x^3 + 242705x^2 + 38970x + 2400\right)$$
$$\rho(30x+18,5) = \tfrac{1}{5!}\left(558930x^5 + 1114290x^4 + 849690x^3 + 307230x^2 + 52260x + 3360\right)$$
$$\rho(30x+19,5) = \tfrac{1}{5!}\left(558930x^5 + 1207445x^4 + 1015740x^3 + 414655x^2 + 81870x + 6240\right)$$
$$\rho(30x+20,5) = \tfrac{1}{5!}\left(558930x^5 + 1300600x^4 + 1181790x^3 + 522080x^2 + 111480x + 9120\right)$$
$$\rho(30x+21,5) = \tfrac{1}{5!}\left(558930x^5 + 1393755x^4 + 1367840x^3 + 659505x^2 + 155890x + 14400\right)$$
$$\rho(30x+22,5) = \tfrac{1}{5!}\left(558930x^5 + 1486910x^4 + 1553890x^3 + 796930x^2 + 200300x + 19680\right)$$
$$\rho(30x+23,5) = \tfrac{1}{5!}\left(558930x^5 + 1580065x^4 + 1758980x^3 + 962435x^2 + 258310x + 27120\right)$$
$$\rho(30x+24,5) = \tfrac{1}{5!}\left(558930x^5 + 1673220x^4 + 1964070x^3 + 1127940x^2 + 316320x + 34560\right)$$
$$\rho(30x+25,5) = \tfrac{1}{5!}\left(558930x^5 + 1766375x^4 + 2204860x^3 + 1358005x^2 + 412550x + 49440\right)$$
$$\rho(30x+26,5) = \tfrac{1}{5!}\left(558930x^5 + 1859530x^4 + 2445650x^3 + 1588070x^2 + 508780x + 64320\right)$$
$$\rho(30x+27,5) = \tfrac{1}{5!}\left(558930x^5 + 1952685x^4 + 2705480x^3 + 1857255x^2 + 631330x + 84960\right)$$
$$\rho(30x+28,5) = \tfrac{1}{5!}\left(558930x^5 + 2045840x^4 + 2965310x^3 + 2126440x^2 + 753880x + 105600\right)$$
$$\rho(30x+29,5) = \tfrac{1}{5!}\left(558930x^5 + 2138995x^4 + 3244180x^3 + 2434745x^2 + 902750x + 132000\right)$$

Additional information about $\rho\text{b}()$ must be acquired to determine values of $\rho\text{b}(x\mathbb{P}_k + b, k)$ as this closed form equation is only for $b = 1$. The counting function $\rho\text{b}()$ is *erratic* as evidenced by the values in Table 3 inducing the requirement of investigating each value of b independently. Equation (7) can be reworked to express $\rho\text{b}()$ with values of $\rho\text{f}()$ by extracting the term for $i = w$ from the summation and reordering the result.

$$\begin{aligned}\rho\text{f}(w,k) &= \sum_{i=k}^{w} \rho\text{b}(i,k) \\ &= \rho\text{b}(w,k) + \sum_{i=k}^{w-1} \rho\text{b}(i,k) \\ &= \rho\text{b}(w,k) + \rho\text{f}(w-1,k)\end{aligned}$$

$$\rho\text{b}(w,k) = \rho\text{f}(w,k) - \rho\text{f}(w-1,k)$$

The table of common possible locations for $\rho\text{b}(x\mathbb{P}_k + 1, k)$ was created from possible location sequences with the restriction that the boundary locations are prime representations. For widths of $x\mathbb{P}_k + 1$ sieved locations a possible location sequence that starts on with a prime representation in the leading boundary location also has a prime representation in the trailing boundary location. This is only true for widths $x\mathbb{P}_k + b$ where $b = 1$. Relieving the restriction so only the leading boundary location is a prime representation creates possible location sequences for the counting function $\rho\text{f}()$. A possible location sequence for $\rho\text{f}(x\mathbb{P}_k + b, k)$ is generated for every prime representation in the sieved locations. The sieved locations are cyclic with a period of $\mathbb{P}_k$ so only the prime representations in the first $\mathbb{P}_k$ sieved locations produce unique possible location sequences. A quantity of $\mathbb{Q}_k$ prime representations exist in the first $\mathbb{P}_k$ sieved locations producing $\mathbb{Q}_k$ possible location sequences.

The $\mathbb{Q}_k$ possible location sequences for $\rho\text{f}(x\mathbb{P}_k + 1, k)$ are the same as those created for $\rho\text{b}(x\mathbb{P}_k + 1, k)$. Removing the trailing boundary location from each possible location sequence of $\rho\text{f}(x\mathbb{P}_k + 1, k)$ creates the possible location sequences for $\rho\text{f}(x\mathbb{P}_k, k)$. The coefficients of the binomials in the combinatorial equations for $\rho\text{f}(x\mathbb{P}_k, k)$ remain the same as those for $\rho\text{b}(x\mathbb{P}_k + 1, k)$ but the terms in the binomials must account for removing the trailing boundary location and relieving the restriction that the trailing boundary location is a prime representation. The binomial set sizes remain the same and the binomial subset sizes are one element larger than the corresponding sizes in the binomials used for $\rho\mathrm{b}(x\mathbb{P}_k+1,k)$. The generated combinatorial equations are converted to polynomials and the coefficients are factored. Finally, the polynomial equations are transformed into a closed form equation.

Combinatorial equations for $\rho\mathrm{f}(x\mathbb{P}_k,k)$.

$$\begin{aligned}
\rho\mathrm{f}(2x,2) &= \tbinom{x-1}{1} \\
\rho\mathrm{f}(6x,3) &= 2\tbinom{2x-1}{2}-\tbinom{x-1}{2} \\
\rho\mathrm{f}(30x,5) &= 8\tbinom{8x-1}{4}-12\tbinom{6x-1}{4}+4\tbinom{4x-1}{4}+6\tbinom{3x-1}{4}-6\tbinom{2x-1}{4}+\tbinom{x-1}{4} \\
\rho\mathrm{f}(210x,7) &= 48\tbinom{48x-1}{6}-120\tbinom{40x-1}{6}-72\tbinom{36x-1}{6}+160\tbinom{32x-1}{6}+180\tbinom{30x-1}{6} \\
&\quad -336\tbinom{24x-1}{6}-60\tbinom{20x-1}{6}+216\tbinom{18x-1}{6}+128\tbinom{16x-1}{6}-90\tbinom{15x-1}{6} \\
&\quad -48\tbinom{12x-1}{6}+90\tbinom{10x-1}{6}-90\tbinom{9x-1}{6}-104\tbinom{8x-1}{6}+144\tbinom{6x-1}{6} \\
&\quad -15\tbinom{5x-1}{6}-20\tbinom{4x-1}{6}-21\tbinom{3x-1}{6}+12\tbinom{2x-1}{6}-\tbinom{x-1}{6}
\end{aligned}$$

Polynomial equations for $\rho\mathrm{f}(x\mathbb{P}_k,k)$ with partially factored coefficients.

$$\begin{aligned}
\rho\mathrm{f}(2x,2) &= \tfrac{1}{1!}\left(\underline{1}x-\underline{1}\right) \\
\rho\mathrm{f}(6x,3) &= \tfrac{1}{2!}\left(7\cdot\underline{1}x^2-3\cdot\underline{3}x+\underline{2}\right) \\
\rho\mathrm{f}(30x,5) &= \tfrac{1}{4!}\left(601\cdot 31\cdot\underline{1}x^4-125\cdot 15\cdot\underline{10}x^3+25\cdot 7\cdot\underline{35}x^2-5\cdot 3\cdot\underline{50}x+\underline{24}\right) \\
\rho\mathrm{f}(210x,7) &= \tfrac{1}{6!}\big(116929\cdot 12265\cdot 127\cdot\underline{1}x^6-16807\cdot 2765\cdot 63\cdot\underline{21}x^5+2401\cdot 601\cdot 31\cdot\underline{175}x^4 \\
&\quad -343\cdot 125\cdot 15\cdot\underline{735}x^3+49\cdot 25\cdot 7\cdot\underline{1624}x^2-7\cdot 5\cdot 3\cdot\underline{1764}x+\underline{720}\big)
\end{aligned}$$

Closed form equation for $\rho\mathrm{f}(x\mathbb{P}_k,k)$.

$$\rho\mathrm{f}(x\mathbb{P}_k,k) = \frac{1}{(k-1)!}\sum_{i=1}^{k-1}\left(s(k,i+1)\,x^i\prod_{j=1}^{\pi(k)}\mathrm{E}_{p_j}^i\right) \tag{20}$$

Again, additional information about $\rho\mathrm{f}()$ must be acquired to determine values of $\rho\mathrm{f}(x\mathbb{P}_k+b,k)$ as the closed form equation is only for $b=1$. The counting function $\rho\mathrm{f}()$ is weakly increasing as evidenced by the equality $\rho\mathrm{f}(2x+2,k)=\rho\mathrm{f}(2x+1,k)$.

Equation (6) can be reworked to express $\rho\text{f}()$ with values of $\rho()$ by extracting the term for $i = w$ from the summation and reordering the result.

$$\begin{aligned}\rho(w,k) &= \sum_{i=k}^{w} \rho\text{f}(i,k) \\ &= \rho\text{f}(w,k) + \sum_{i=k}^{w-1} \rho\text{f}(i,k) \\ &= \rho\text{f}(w,k) + \rho(w-1,k)\end{aligned}$$

$$\rho\text{f}(w,k) = \rho(w,k) - \rho(w-1,k)$$

Using the method that created the equations for $\rho\text{f}(x\mathbb{P}_k, k)$ the equations for $\rho(x\mathbb{P}_k - 1, k)$ can be created by relieving the restriction that the leading boundary location is a prime representation. Possible location sequences for $\rho(x\mathbb{P}_k - 1, k)$ are generated at every sieved location. The sieved locations are cyclic with a period of $\mathbb{P}_k$ this time producing a total of $\mathbb{P}_k$ possible location sequences. Of these $\mathbb{P}_k$ possible location sequences there are $\mathbb{Q}_k$ sequences that have a prime representation in the leading boundary.

The $\mathbb{Q}_k$ possible location sequences that have a prime representation in the leading boundary are the same as those created for $\rho\text{f}(x\mathbb{P}_k, k)$. The difference again occurs in the combinatorial equations. The coefficients of the binomials in the combinatorial equations for $\rho(x\mathbb{P}_k - 1, k)$ remain the same as those for $\rho\text{f}(x\mathbb{P}_k, k)$ but the terms in the binomials must account removing the leading boundary location and relieving the restriction that the leading boundary is a prime representation. The binomial set sizes remain the same and the binomial subset sizes are one element larger than the corresponding sizes in the binomials used for $\rho\text{f}(x\mathbb{P}_k, k)$.

The combinatorial equations that represent the remaining $\mathbb{P}_k - \mathbb{Q}_k$ possible location sequences have binomial set sizes that are one element smaller while the binomial subset sizes remain the same. The generated combinatorial equations are converted to polynomials and the coefficients are factored. Finally, the polynomial equations are transformed into a closed form equation.

Combinatorial equations for $\rho(x\mathbb{P}_k - 1, k)$.

$$\begin{aligned}\rho(2x-1,2) &= \tbinom{x}{2} + \tbinom{x-1}{2} \\ \rho(6x-1,3) &= 4\tbinom{2x}{3} + 2\tbinom{2x-1}{3} - 5\tbinom{x}{3} - \tbinom{x-1}{3} \\ \rho(30x-1,5) &= 22\tbinom{8x}{5} + 8\tbinom{8x-1}{5} - 48\tbinom{6x}{5} - 12\tbinom{6x-1}{5} \\ &\quad +26\tbinom{4x}{5} + 4\tbinom{4x-1}{5} + 54\tbinom{3x}{5} + 6\tbinom{3x-1}{5} \\ &\quad -24\tbinom{2x}{5} - 6\tbinom{2x-1}{5} + 29\tbinom{x}{5} + \tbinom{x-1}{5}\end{aligned}$$

fill in equations

Polynomial equations for $\rho(x\mathbb{P}_k - 1, k)$ with partially factored coefficients.

$$\begin{aligned}
\rho(2x-1,2) &= \tfrac{1}{2!}\left(xxx \right) \\
\rho(6x-1,3) &= \tfrac{1}{3!}\left(xxx \right) \\
\rho(30x-1,5) &= \tfrac{1}{5!}\big(xxx \\
&\qquad xxx \big)
\end{aligned}$$

verify equation

Closed form equation for $\rho(x\mathbb{P}_k - 1, k)$.

$$\rho(x\mathbb{P}_k - 1, k) = \frac{1}{k!}\sum_{i=2}^{k}\left(s(k+1, i+1)\, x^i \prod_{j=1}^{\pi(k)} \mathrm{E}_{p_j}^{i} \right) \tag{21}$$

continue here

Generalize the closed form equation for $\rho(x\mathbb{P}_k + b, k)$

Generalize the closed form equation for $\rho\mathrm{f}(x\mathbb{P}_k + b, k)$

Generalize the closed form equation for $\rho\mathrm{b}(x\mathbb{P}_k + b, k)$